THE APPLICATION OF THE DISCRETE MAXIMUM PRINCIPLE TO TRANSPORTATION PROBLEMS WITH LINEAR AND NON-LINEAR COST FUNCTIONS

by

JAGDISHCHANDRA MAGANLAL PANCHAL

B. E. (Mech.) Maharaja Sayajirao University
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Approved by:

Ching-Lai Hwang
Major Professor

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INTRODUCTION

Optimization of transportation problems with linear cost functions can be regarded as a generalization of the assignment problems and can be accomplished by the Simplex Method of Linear Programming [1]. However, some special methods, such as the Northwest Corner Method, the Unit Penalty Method and Vogel's Approximation Method, have been developed which are easy to apply and are less tedious than the Simplex Method [2, 3]. Recently, a discrete version of the Maximum Principle has been applied to the two, three and four depots problems with ease, in view of calculations [4].

Optimization of transportation problems with non-linear cost functions can no longer be solved by Linear Programming Methods. Such problems for two and three depots are solved by Dynamic Programming [5]. Recently, a discrete version of the Maximum Principle has been applied to the two depots problem. This has resulted in a great simplification of numerical calculations [6, 7].

The three depots problem with non-linear cost function is investigated by Hwang, et al [8]. The Maximum Principle for continuous processes was originally developed by Pontryagin [9]. The Discrete version of this Maximum Principle was proposed by Chang [10] and Katz [11] and was developed further by Fan and Wang [6].

The aim of this report is to present the application of the Discrete Maximum Principle to obtain the solution of transportation problems having both linear and non-linear cost functions in a relatively elegant manner. Simple problems of the linear type with two and three origins are systematically analyzed in order to develop a generalized computational procedure for solving problems having more than three origins. A problem with four

origins is solved to illustrate in detail both this general computational procedure and a systematic search for feasible solutions and then an optimal solution. Simple problems of two and three depots having non-linear cost functions, with and without set-up costs, are also systematically illustrated.

Very recently, Charnes and Kortanek [12] have commented on the Discrete Maximum Principle and Distributation Problems published by Fan and Wang [7]. The simple example for a linear cost function cited by Charnes and Kortanek is included in this report. The systematic search for an optimal solution is applied to demonstrate that their comment on having serious difficulty with numerical procedures to obtain an optimum solution is premature. As this method is in an early stage of development, it does not appear to be appropriate to compare the efficiency of this present method with that of others which have been more fully refined.

THE DISCRETE MAXIMUM PRINCIPLE

The following is an outline of the general algorithm of the Discrete Maximum Principle for systems without information feedback given by Fan and Wang [6].

A multistage decision process consisting of N-stages in sequence is schematically shown in Fig. 1. The state of the process stream, denoted by an s-dimensional vector, x, is transformed at each stage according to the decision made on the control actions denoted by a t-dimensional vector, θ. The transformation of the process stream thus brought about at the n^{th} stage is given by the transformation operator (or performance equation)

$$x_i^n = T_i^n \,(x_1^{n-1}, x_2^{n-1}, \ldots, x_s^{n-1}; \theta_1^n, \theta_2^n, \ldots, \theta_t^n). \qquad (1)*$$

$$n = 1, 2, \ldots, N; \; i = 1, 2, \ldots, s.$$

or, in vector form,

$$x^n = T^n \,(x^{n-1}; \theta^n).$$

The optimization problem is to determine the sequence of θ^n, subject to the constraints, $\eta^n \leq \theta^n \leq \xi^n$, $n = 1, 2, \ldots, N$, which will maximize $\sum_{i=1}^{s} c_i x_i^n$, with x_i^o preassigned, $i = 1, 2, \ldots, s$. Here η^n and ξ^n are the lower and upper bounds of θ^n and c_i denotes some specified constants.

The procedure for finding the optimal sequence of θ^n is to introduce an adjoint vector, z^n, and a Hamiltonian function, H^n, satisfying

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* The superscript, n, indicates the stage number. The exponents are written with parentheses or brackets such as $(x^n)^2$ or $\{\phi(x^n)\}^2$.

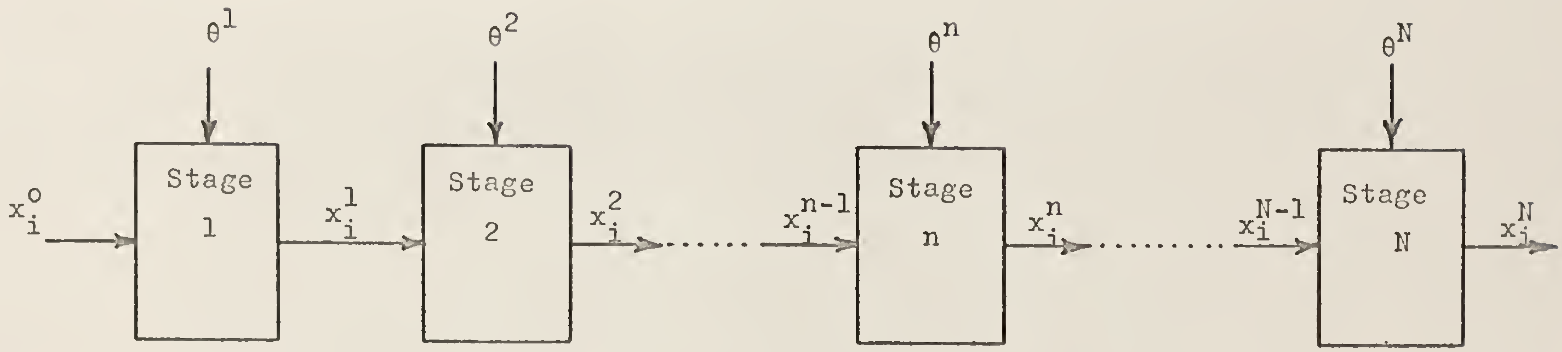

Fig. 1. Multistage decision process.

$$H^n = \sum_{i=1}^{s} z_i^n T_i^n (x^{n-1}; \theta^n), \qquad n = 1, 2, \ldots, N. \tag{2}$$

$$z_i^{n-1} = \frac{\partial H^n}{\partial x_i^{n-1}}, \qquad n = 1, 2, \ldots, N; \; i = 1, 2, \ldots, s. \tag{3}$$

$$z_i^n = c_i, \qquad i = 1, 2, \ldots, s. \tag{4}$$

and to determine the optimal sequence of control actions, $\bar{\theta}^n$, from the conditions

$$H^n = \text{maximum, or} \quad \frac{\partial H^n}{\partial \theta^n} = 0. \tag{5}$$

For the optimization problem in which some of the final values of state variables x_i^N are preassigned, such as $x_a^N = W_a$ and $x_b^N = W_b$, and the objective function is specified as

$$\sum_{\substack{i=1 \\ i \neq a \\ i \neq b}}^{s} c_i \, x_i^n$$

the basic algorithm represented by equations (2) through (5) is still applicable, except that equation (4) is replaced by

$$z_i^n = c_i \qquad \begin{cases} i = 1, 2, \ldots, s. \\ i \neq a, b. \end{cases} \tag{6}$$

If the minimizing, instead of the maximizing, sequence of control actions is to be decided, the above algorithm remains applicable, except that equation (5) is replaced by

$$H^n = \text{minimum, or} \quad \frac{\partial H^n}{\partial \theta^n} = 0, \qquad n = 1, 2, \ldots, N. \tag{7}$$

FORMULATION OF THE TRANSPORTATION PROBLEM BY THE DISCRETE MAXIMUM PRINCIPLE

The transportation problems having linear as well as non-linear cost functions shall first be formulated in terms of the discrete maximum principle.

Suppose that there is only one type of resource and that its total supply is equal to the total demand for it. Let

θ_i^n = the quantity of the resource sent from the i-th depot (origin) to the n-th demand point and

$F_i^n(\theta_i^n)$ = the cost incurred by this operation.

If there are s depots and N demand points, the problem is to determine the values of θ_i^n, $i = 1, 2, \ldots, s$; $n = 1, 2, \ldots, N$, so as to minimize the total cost of transporting the resource

$$c_{sN} = \sum_{n=1}^{N} \sum_{i=1}^{s} F_i^n(\theta_i^n)$$

subject to the constraints

(i) $\theta_i^n \geq 0$

(ii) $\sum_{n=1}^{N} \theta_i^n = W_i$, number of units of the resource available at the i-th depot, $i = 1, 2, \ldots, s$.

(iii) $\sum_{i=1}^{s} \theta_i^n = D^n$, number of units of the resource required by the n-th demand point, $n = 1, 2, \ldots, N$.

Defining the demand points as stages and the total amount of resource which has been transported from the i-th depot to the first n stages (demand points) as state variables x_i^n, $i = 1, 2, \ldots, s-1$, then

$$x_i^n = x_i^{n-1} + \theta_i^n, \qquad x_i^o = 0, \qquad x_i^N = W_i$$

$$i = 1, 2, \ldots, s-1, \quad n = 1, 2, \ldots, N.$$

It must be noted that, though there are "s" depots in the problem, there are only (s-1) state variables. This is because the demand by each stage is preassigned; hence the number of the units supplied from the s-th depot to n-th stage can be obtained by subtracting the sum of the units supplied to the n-th stage from the first through (s-1)-th depots from the total number of units required by the n-th stage. That is

$$\theta_s^n = D^n - \sum_{i=1}^{s-1} \theta_i^n$$

Since it is desired to minimize the total cost of transportation, a new state variable, x_s^n, may be defined as

$$\begin{aligned} x_s^n &= x_s^{n-1} + \sum_{i=1}^{s} F_i^n (\theta_i^n) \\ x_s^o &= 0, \qquad n = 1, 2, \ldots, N. \end{aligned} \tag{9}$$

It can be shown that x_s^N is equal to the total cost of transportation. The optimization problem is formulated as one in which x_s^N is to be minimized by the proper choice of the sequence of θ_i^n, $i = 1, 2, \ldots, s-1$, $n = 1, 2, \ldots, N$, for the process described by equations (8) and (9).

A discrete version of the maximum principle asserts that, for finding the optimal sequence of θ^n, if the adjoint vector, z^n, and the Hamiltonian function, H^n, satisfying

$$H^n = \sum_{i=1}^{s} z_i^n x_i^n (x^{n-1}; \theta^n), \qquad n = 1, 2, \ldots, N \tag{10}$$

$$z_i^{n-1} = \frac{\partial H^n}{\partial x_i^{n-1}}, \qquad \begin{aligned} n &= 1, 2, \ldots, N \\ i &= 1, 2, \ldots, s \end{aligned} \tag{11}$$

$$z_s^N = 1 \tag{12}$$

are introduced, and the optimal sequence of $\bar{\theta}^n$ is obtained from the condition

$$H^n = \begin{cases} \text{stationary at the interior point of } \theta^n \\ \text{minimum at the boundary point of } \theta^n \end{cases}$$

$$n = 1, 2, \ldots, N.$$

For the process under consideration, the Hamiltonian function can be written as

$$H^n = \sum_{i=1}^{s-1} z_i^n (x_i^{n-1} + \theta_i^n) + z_s^n \left\{ x_s^{n-1} + \sum_{i=1}^{s} F_i^n (\theta_i^n) \right\} \tag{13}$$

$$n = 1, 2, \ldots, N.$$

and components of the adjoint vector are, in general,

$$z_i^{n-1} = \frac{\partial H^n}{\partial x_i^{n-1}} = z_i^n , \qquad i = 1, 2, \ldots, s \tag{14}$$

Equation (12) results specifically in

$$z_s^n = 1, \qquad n = 1, 2, \ldots, N.$$

Since z_i^n and x_i^{n-1} are considered as constants at each step in the minimization of the Hamiltonian function given by equation (13), it is convenient to define the variable part of the Hamiltonian function as

$$H_v^n = \sum_{i=1}^{s-1} z_i^n \theta_i^n + \sum_{i=1}^{s} F_i^n (\theta_i^n). \tag{15}$$

EXAMPLE (1). TWO ORIGINS AND FOUR DEMAND POINTS (LINEAR COST FUNCTION)

The linear cost function, $F_i^n(\theta_i^n)$ can be expressed by

$$F_i^n(\theta_i^n) = C_i^n \theta_i^n$$

where

C_i^n = the cost incurred in supplying one unit of resource from the i-th origin to the n-th demand point.

The problem is represented by Table 1. Values of C_i^n (in dollars), D^n and W_i are shown in this table. The total number of units required by N-demand points is equal to the total number of units supplied from the s-origins, that is,

$$\sum_{n=1}^{N} D^n = \sum_{i=1}^{s} W_i.$$

It is required to allocate the number of resource units in such a way as to minimize the total cost of transportation.

Table 1. Transportation costs and requirements for Example (1).

Depots

Demand points: n \ i	1	2	D^n
1	8	3	8
2	5	8	20
3	1	3	12
4	7	2	5
W_i	25	20	45

The variable part of the Hamiltonian equation for this problem is Equation (15)

$$H_v^n = z_1^n\theta_1^n + \sum_{i=1}^{2} C_i^n\theta_i^n$$

$$= z_1^n\theta_1^n + C_1^n\theta_1^n + C_2^n\theta_2^n, \qquad n = 1, 2, 3, 4.$$

Since $\theta_2^n = D^n - \theta_1^n$, the following is obtained

$$H_v^n = (z_1^n + C_1^n - C_2^n)\ \theta_1^n + C_2^n D^n, \qquad n = 1, 2, 3, 4.$$

Stage 1:

Substituting n = 1 in the foregoing equation, the variable part of the Hamiltonian equation for the first demand point (stage) becomes

$$H_v^1 = (z_1^1 + C_1^1 - C_2^1)\ \theta_1^1 + C_2^1 D^1.$$

From the entries in Table 1, this becomes

$$H_v^1 = (z_1^1 + 5)\ \theta_1^1 + 24.$$

Thus,

$$\bar{z}_1^1 = -5 = C_2^1 - C_1^1.$$

From this three conditions at which H_v^1 may be minimum result:

(a) H_v^1 = min. at $\theta_1^1 = 0$ when $z_1^1 > -5$

(b) H_v^1 = min. at $0 \leq \theta_1^1 \leq 8$ when $z_1^1 = -5$

(c) H_v^1 = min. at $\theta_1^1 = 8$ when $z_1^1 < -5$.

The conditions (a), (b) and (c) are shown in Fig. 2.

In a similar manner, the values of z_1^n and θ_1^n are determined for the rest of the demand points (stages), n = 2, 3 and 4, which makes H_v^n a minimum.

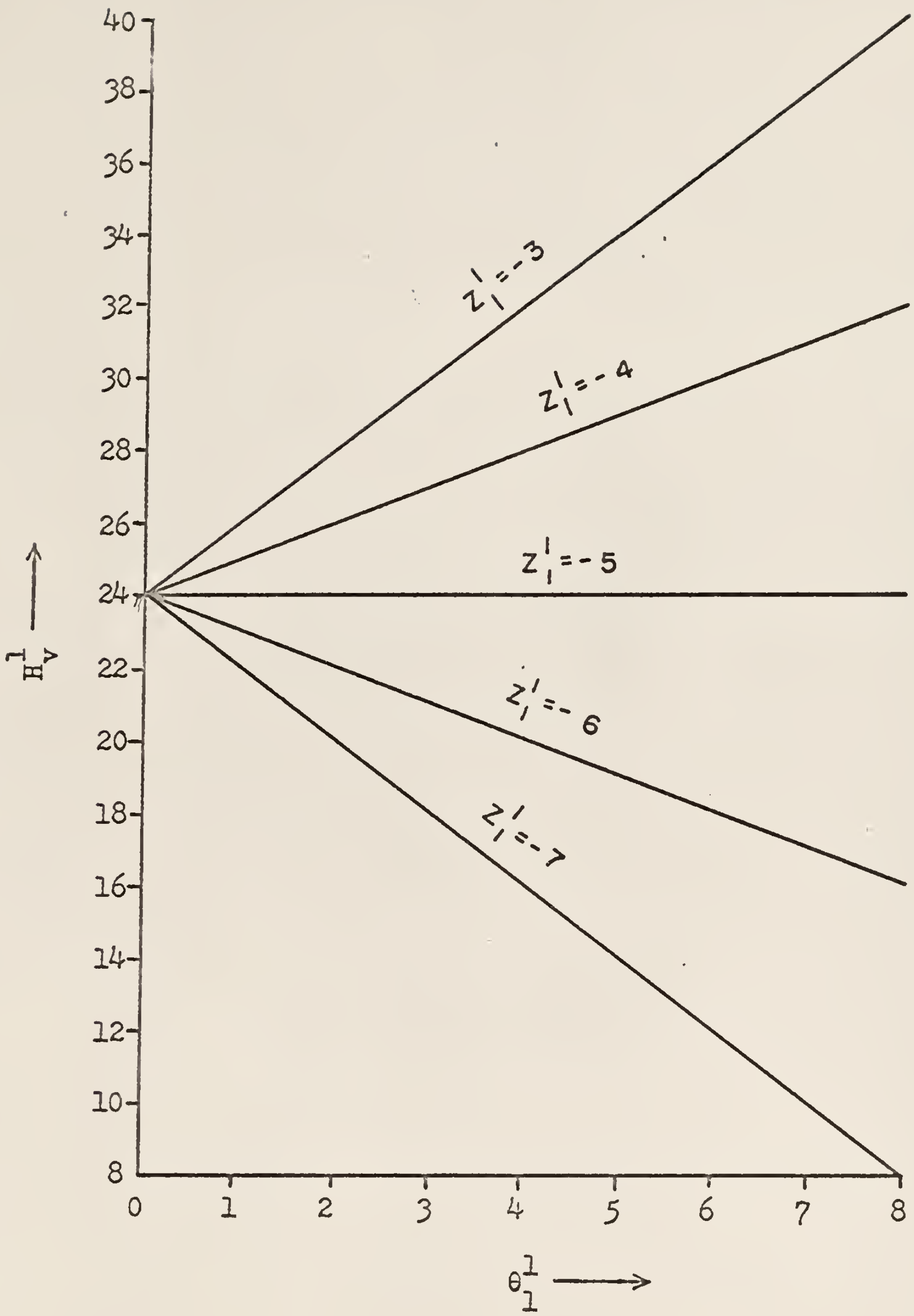

Fig. 2. Adjoint vector z_1^1, showing selection of θ_1^1 for Example (1).

These values of z_1^n and θ_1^n are shown in Table 2.

Table 2. Conditions necessary for H_v^n to be minimum for Example (1).

n	Minima of H_v^n occuring at	
	θ_1^n	z_1^n
1	0	> -5
	$0 \leq \theta_1^1 \leq 8$	$= -5$
	8	< -5
2	0	> 3
	$0 \leq \theta_1^2 \leq 20$	$= 3$
	20	< 3
3	0	> 2
	$0 \leq \theta_1^3 \leq 12$	$= 2$
	12	< 2
4	0	> -5
	$0 \leq \theta_1^4 \leq 5$	$= -5$
	5	< -5

As given by Equation (14), the value of z_1^n, $n = 1, 2, 3, 4$ are identical. From the values of z_1^n, $n = 1, 2, 3$ and 4 given in Table 2, Fig. 3 shows the boundary values of z_1^n, i.e., $\bar{z}_1^n$.

First, the value of z_1 which gives all solutions satisfying the constraints given by conditions (i), (ii), and (iii) will be obtained; then,

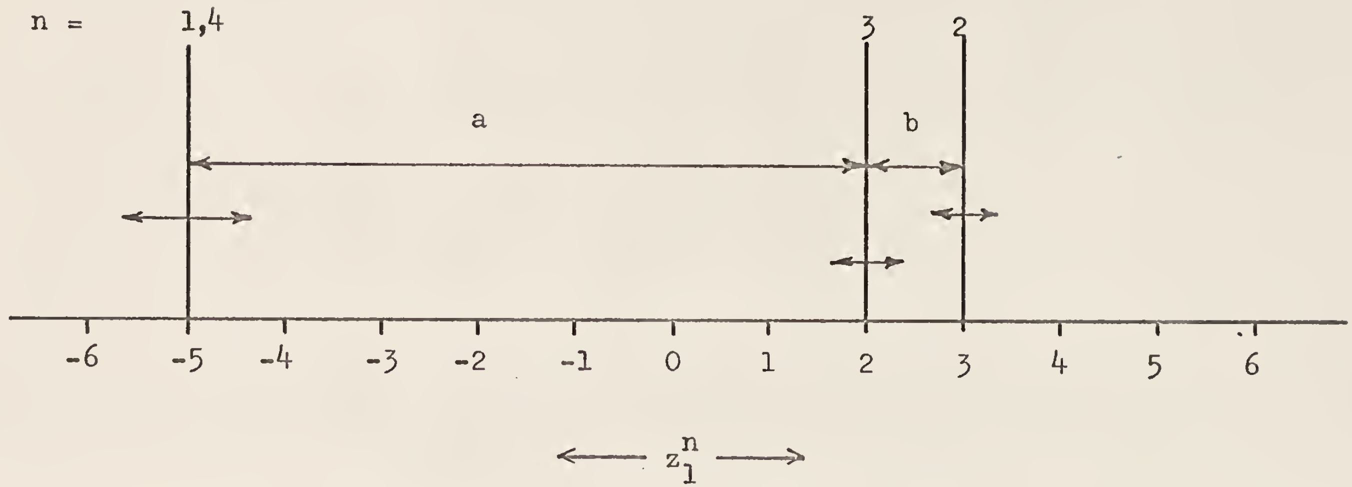

Fig. 3. Boundary values of adjoint vector z_1^n for Example (1).

the solution which minimizes the cost, that is, an optimal solution, will be chosen. For illustration, the solutions corresponding to the values of x_1^n in the region of $-5 < z_1^n < 2$, $n = 1, 2, 3$ and 4 will be given. Comparing the values of z_1^n, $n = 1, 2, 3, 4$ shown in Table 2 with the boundary values of z_1^n which define the region, the values of θ_1^n given in Table 3 can be obtained. This solution does not, however, satisfy the end-point condition of $W_1 = 25$.

Table 3. θ_i^n corresponding to the values of z_1^n in the region of $-5 < z_1^n < 2$.

n \ i	1	2	D^n
1	0	8	8
2	20	0	20
3	12	0	12
4	0	5	5
W_i	32 (25)	13 (20)	45

Then the corresponding solution of $z_1^n = 2$ is found. The results, summarized in Table 4, give the feasible solution which satisfies the end-point conditions.

In order to satisfy the end-point conditions, $W_1 = 25$ and $W_2 = 20$, θ_1^3 has to be 5. The total cost for this solution is

$$\sum_{n=1}^{4} \sum_{i=1}^{2} C_i^n \theta_i^n = \$\ 160.$$

This is the only feasible solution and should be the optimal solution. The solution obtained by the linear programming method is the same.

Table 4. θ_i^n corresponding to the values of z_1^n at $z_1^n = 2$

$n \backslash i$	1	2	D^n
1	0	8	8
2	20	0	20
3	(5) $0 \leq \theta_1^3 \leq 12$	(7) $12 - \theta_1^3$	12
4	0	5	5
W_i	$20 + \theta_1^3$ (25)	$25 - \theta_1^3$ (20)	45

EXAMPLE (2). THREE ORIGINS AND FOUR DEMAND POINTS (LINEAR COST FUNCTION)

The problem is represented by Table 5.

Table 5. Transportation costs and requirements for Example (2).

Demand points	Depots			
n \ i	1	2	3	D^n
1	8	7	4	18
2	5	8	1	29
3	2	6	2	23
4	4	3	3	25
W_i	20	30	45	95

The variable part of the Hamiltonian equation for this problem is

$$H_v^n = \sum_{i=1}^{2} z_i^n \theta_i^n + \sum_{i=1}^{3} c_i^n \theta_i^n , \qquad n = 1, 2, 3, 4.$$

$$= z_1^n \theta_1^n + z_2^n \theta_2^n + C_1^n \theta_1^n + C_2^n \theta_2^n + C_3^n \theta_3^n$$

Since $\theta_3^n = D^n - \theta_1^n - \theta_2^n$, then

$$H_v^n = (z_1^n + C_1^n - C_3^n)\, \theta_1^n + (z_2^n + C_2^n - C_2^n)\, \theta_2^n + C_3^n D^n$$

Stage 1:

The variable part of the Hamiltonian equation for the first demand point (stage) is

$$H_V^1 = (z_1^1 + C_1^1 - C_3^1)\,\theta_1^1 + (z_2^1 + C_2^1 - C_3^1)\,\theta_2^1 + C_3^1\,D^1$$

From the entries in Table 1, this becomes

$$H_V^1 = (z_1^1 + 4)\,\theta_1^1 + (z_2^1 + 3)\,\theta_2^1 + 72$$

Thus

$$\bar{z}_1^1 = -4 = C_3^1 - C_1^1\,, \qquad \bar{z}_2^1 = -3 = C_3^1 - C_2^1$$

From this nine conditions at which H_V^1 may be minimum result:

(a) $H_V^1 = \min.$ at $\theta_1^1 = 0$ & $\theta_2^1 = 0$ when $z_1^1 > -4$ & $z_2^1 > -3$

(b) $H_V^1 = \min.$ at $0 \leq \theta_1^1 \leq 18$ & $\theta_2^1 = 0$ when $z_1^1 = -4$ & $z_2^1 > -3$

(c) $H_V^1 = \min.$ at $\theta_1^1 = 18$ & $\theta_2^1 = 0$ when $z_1^1 < -4$ & $z_2^1 > -3$

(d) $H_V^1 = \min.$ at $\theta_1^1 = 0$ & $0 \leq \theta_2^1 \leq 18$ when $z_1^1 > -4$ & $z_2^1 = -3$

(e) $H_V^1 = \min.$ at $0 \leq \theta_1^1 \leq 18$ & $0 \leq \theta_2^1 \leq 18$ when $z_1^1 = -4$ & $z_2^1 = -3$

(f) $H_V^1 = \min.$ at $\theta_1^1 = 18$ & $\theta_2^1 = 0$ when $z_1^1 < -4$ & $z_2^1 = -3$

(g) $H_V^1 = \min.$ at $\theta_1^1 = 0$ & $\theta_2^1 = 18$ when $z_1^1 > -4$ & $z_2^1 < -3$

(h) $H_V^1 = \min.$ at $\theta_1^1 = 0$ & $\theta_2^1 = 18$ when $z_1^1 = -4$ & $z_2^1 < -3$

(i) $H_V^1 = \min.$ at $0 \leq \theta_1^1 \leq 18$ & $0 \leq \theta_2^1 \leq 18$ when $z_1^1 < -4$ & $z_2^1 < -3$.

In a similar manner, the values of z_1^n, z_2^n, θ_1^n, and θ_2^n are determined for the rest of the demand points (stages), n = 2, 3 and 4, which makes H_V^n a minimum. These values of z_1^n, z_2^n, θ_1^n and θ_2^n are shown in Table 6.

Table 6. Conditions necessary for H_v^n to be minimum for Example (2)

n	Minimum of H_v^n occurs at			
	θ_1^n	θ_2^n	z_1^n	z_2^n
1	0	0	> -4	> -3
	$0 \leq \theta_1^1 \leq 18$	0	$= -4$	> -3
	18	0	< -4	> -3
	0	$0 \leq \theta_2^1 \leq 18$	> -4	$= -3$
	$0 \leq \theta_1^1 \leq 18$	$0 \leq \theta_2^1 \leq 18$	$= -4$	$= -3$
	18	0	< -4	$= -3$
	0	18	> -4	< -3
	0	18	$= -4$	< -3
	$0 \leq \theta_1^1 \leq 18$	$0 \leq \theta_2^1 \leq 18$	< -4	< -3
2	0	0	> -4	> -7
	$0 \leq \theta_1^2 \leq 29$	0	$= -4$	> -7
	29	0	< -4	> -7
	0	$0 \leq \theta_2^2 \leq 29$	> -4	$= -7$
	$0 \leq \theta_1^2 \leq 29$	$0 \leq \theta_2^2 \leq 29$	$= -4$	$= -7$
	29	0	< -4	$= -7$
	0	29	> -4	< -7
	0	29	$= -4$	< -7
	$0 \leq \theta_1^2 \leq 29$	$0 \leq \theta_2^2 \leq 29$	< -4	< -7

Table 6. (Continued)

n	Minimum of H_v^n occurs at			
	θ_1^n	θ_2^n	z_1^n	z_2^n
3	0	0	> 0	> -4
	$0 \leq \theta_1^3 \leq 23$	0	$= 0$	> -4
	23	0	< 0	> -4
	0	$0 \leq \theta_2^3 \leq 23$	> 0	$= -4$
	$0 \leq \theta_1^3 \leq 23$	$0 \leq \theta_2^3 \leq 23$	$= 0$	$= -4$
	23	0	< 0	$= -4$
	0	23	> 0	< -4
	0	23	$= 0$	< -4
	$0 \leq \theta_1^3 \leq 23$	$0 \leq \theta_2^3 \leq 23$	< 0	< -4
4	0	0	> -1	> 0
	$0 \leq \theta_1^4 \leq 25$	0	$= -1$	> 0
	25	0	< -1	> 0
	0	$0 \leq \theta_2^4 \leq 25$	> -1	$= 0$
	$0 \leq \theta_1^4 \leq 25$	$0 \leq \theta_2^4 \leq 25$	$= -1$	$= 0$
	25	0	< -1	$= 0$
	0	25	> -1	< 0
	0	25	$= -1$	< 0
	$0 \leq \theta_1^4 \leq 25$	$0 \leq \theta_2^4 \leq 25$	< -1	< 0

The conditions for all H_v^n to be minimum are tabulated in Table 6. As given by Equation (14), z_1^1, z_1^2, z_1^3 and z_1^4 are identical. Similarly, z_2^1, z_2^2, z_2^3 and z_2^4 are also identical. From the values of z_1^n and z_2^n given in Table 6, Figs. 4a and 4b show the boundary values of z_1^n and z_2^n; i.e.; $\bar{z}_1^n$ and $\bar{z}_2^n$.

By systematic search of each combination of the interior and/or boundary values of z_1^n and z_2^n (see Figs. 4a and 4b) for feasible solutions, cases which do not satisfy the constraints are eliminated. For instance, the value z_1^n in the region $z_1^n > 0$, together with any values of z_2^n will yield $\sum_{n=1}^{4} \theta_1^n = 0$, which does not satisfy the constraint $W_1 = 20$. Similarly the combination of $z_1^n = 0$ and $z_2^n = 0$; $z_1^n = 0$ and $z_2^n = -4$; $z_1^n = -1$ and $z_2^n = 0$; $z_1^n = -1$ and $z_2^n = -3$ etc., does not give feasible solution except that $z_1^n = 0$ and $z_2^n = -3$. For example, the values of θ_i^n corresponding to the values of $z_1^n = -1$ and $z_2^n = -3$ are presented in Table 7.

Table 7. θ_i^n corresponding to the values of $\bar{z}_1^n = -1$ and $\bar{z}_2^n = -3$.

n \ i	1	2	3	D^n
1	0	$0 \leq \theta \leq 18$	0	18
2	0	0	29	29
3	23	0	0	23
4	0	25	0	25
W_i	(20)	(30)	(45)	95

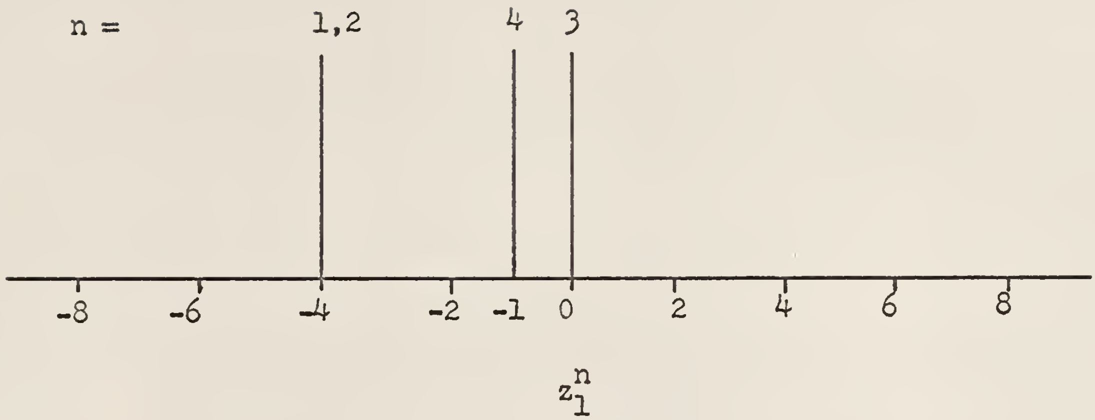

Fig. 4a. Boundary values of adjoint vector z_1^n for Example (2).

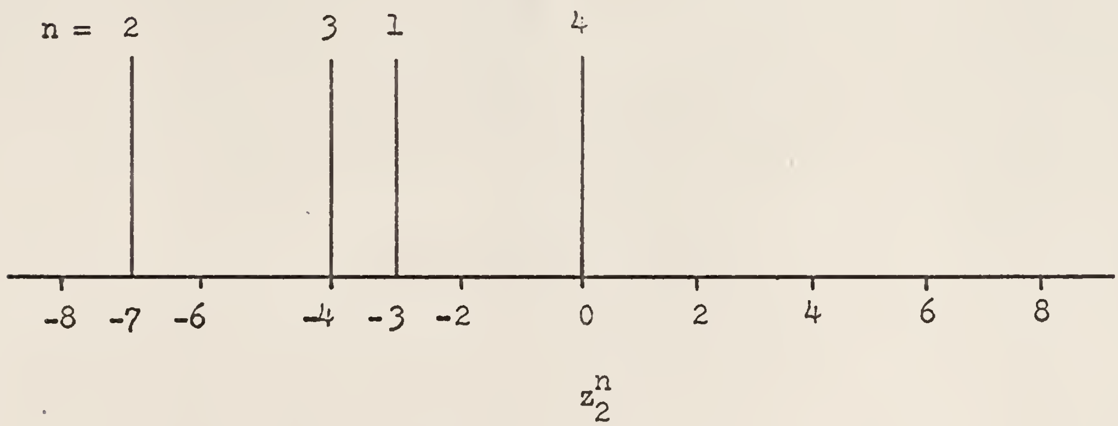

Fig. 4b. Boundary values of adjoint vector z_2^n for Example (2).

Again, the constraint of $W_1 = 20$ cannot be satisfied.

The results at $z_1^n = 0$ and $z_2^n = -3$ are shown in Table 8.

Table 8. θ_i^n corresponding to the values of z_1^n at $z_1^n = 0$ and $z_2^n = -3$.

n \ i	1	2	3	D^n
1	0	$0 \leq \theta_2^1 \leq 18$	$18 - \theta_2^1$	18
2	0	0	29	29
3	$0 \leq \theta_1^3 \leq 23$	0	$23 - \theta_1^3$	23
4	0	25	0	25
W_i	θ_1^3 (20)	$25 + \theta_2^1$ (30)	$70 - (\theta_2^1 + \theta_1^3)$ (45)	95

From Table 8, it can be seen that $\theta_2^1 = W_2 - 25 = 30 - 25 = 5$, and $\theta_1^3 = 20$. The solution is presented in Table 9.

The total cost for the solution shown in Table 9 is

$$\sum_{n=1}^{4} \sum_{i=1}^{3} C_i^n \theta_i^n = \$337.$$

This is the only feasible solution, therefore, the optimal solution. The solution obtained by the linear programming method is the same.

Table 9. The solution for $z_1^n = 0$ and $z_2^n = -3$.

n \ i	1	2	3	D^n
1	0	(5)	(13)	18
2	0	0	29	29
3	(20)	0	(3)	23
4	0	25	0	25
W_i	20	30	45	95

COMPUTATIONAL PROCEDURES FOR PROBLEMS WITH LINEAR COST FUNCTION

The computational procedures for problems with linear-cost function may be developed and summarized as follows:

Since the linear cost function is

$$F_i^n(\theta_i^n) = C_i^n \theta_i^n$$

the variable part of the Hamiltonian function given by Equation (15) is

$$H_V^n = \sum_{i=1}^{s-1} z_i^n \theta_i^n + \sum_{i=1}^{s} C_i^n \theta_i^n, \quad n = 1, 2, \ldots, N. \tag{16}$$

Since H^n is linear in θ_i^n, the values of θ_i^n are determined in such a way the H^n is absolute minimum. It should be noted that, since z_i^n is undetermined at the beginning of calculation, it plays a role similar to the Lagrange multiplier in differential calculus. The values of z_i^n are to be determined at the end of calculation from the condition that the values of x_i^N are W_i.

Equation (16) can be written in the form

$$H_V^n = \sum_{i=1}^{s-1} (z_i^n + C_i^n)\, \theta_i^n + C_s^n \theta_s^n \tag{17}$$

Substituting $\theta_s^n = D^n - \sum_{i=1}^{s-i} \theta_i^n$ into this equation yields

$$H_V^n = \sum_{i=1}^{s-1} (z_i^n + C_i^n)\, \theta_i^n + C_s^n \left(D^n - \sum_{i=1}^{s-1} \theta_i^n\right)$$

$$= \sum_{i=1}^{s-1} \left\lfloor z_i^n + (C_i^n - C_s^n) \right\rfloor \theta_i^n + C_s^n D^n \tag{18}$$

Since $C_s^n D^n$ is constant, values of θ_i^n, which give the minimum H_V^n, depend on the sign of the bracketed quantity $\left\lfloor z_i^n + (C_i^n - C_s^n) \right\rfloor$. For any i, the value

of z_i^n, at which this bracketed quantity changes its sign, may be called the boundary value of z_i^n. It is located where

$$z_i^n + (C_i^n - C_s^n) = 0$$

or

$$\bar{z}_i^n = C_s^n - C_i^n \tag{19}$$

In addition, the three constraints are

(i) $\theta_i^n \geq 0$

(ii) $\sum_{i=1}^{s} \theta_i^n = D^n$

(iii) $\sum_{n=1}^{N} \theta_i^n = x_i^N = W_i$

Therefore, based on Equations (18) and (19) and constraints (i), (ii) and (iii), the computational procedure may be summarized:

(1) For any particular value of z_i^n considered,

(a) If, for any i, $z_i^n > \bar{z}_i^n$, then $\theta_i^n = 0$

(b) If, for any i, $z_i^n < \bar{z}_i^n$, then θ_i^n is a positive value such that $0 \leq \theta_i^n \leq D^n$. And, as a special case, if $z_j^n < \bar{z}_j^n$ for only one j, and $z_i^n \geq \bar{z}_i^n$ for all $i \neq j$, then θ_j^n takes the extreme value, that is $\theta_j^n = D^n$.

(c) If, for any i, $z_i^n = \bar{z}_i^n$, then the corresponding θ_i^n is such that $0 \leq \theta_i^n \leq D^n$ except the special case mentioned in (b).

(2) Then all the values of z_i^n and eventually θ_i^n are fixed from Equation (11) and constraints (ii) and (iii).

(3) Finally, an optimal solution or solutions which give the minimum cost function are selected from all the resulting feasible solutions.

EXAMPLE (3). FOUR ORIGINS AND FIVE DEMAND POINTS (LINEAR COST FUNCTION)

The problem is represented by Table 10.

Table 10. Transportation costs and requirements for Example (3)

Depots

Demand points: n \ i	1	2	3	4	D^n
1	3	3	6	4	10
2	5	2	10	9	20
3	5	7	3	8	10
4	4	10	2	10	18
5	8	3	3	12	20
W_i	16	20	18	24	78

The variable part of the Hamiltonian equation for this problem is [equation (18)]

$$H_v^n = \sum_{i=1}^{3} \left\lfloor z_i^n + (C_i^n - C_4^n) \right\rfloor \theta_i^n + C_4^n D^n, \qquad n = 1, 2, 3, 4, 5.$$

The boundary values of z_i^n, i.e., $\bar{z}_i^n$, i = 1, 2, 3, obtained by the use of Equation (19), are plotted on Figs. 5a, 5b and 5c and are listed in Table 11.

The systematic search for feasible solutions combining the interior and/or boundary values of z_i^n (see Figs. 5a, 5b, and 5c) is as follows:

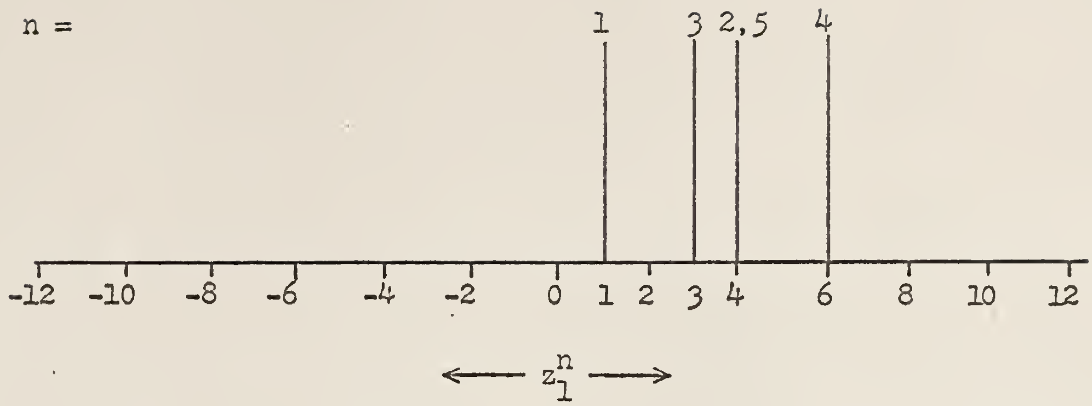

Fig. 5a. Boundary values of adjoint vector z_1^n.

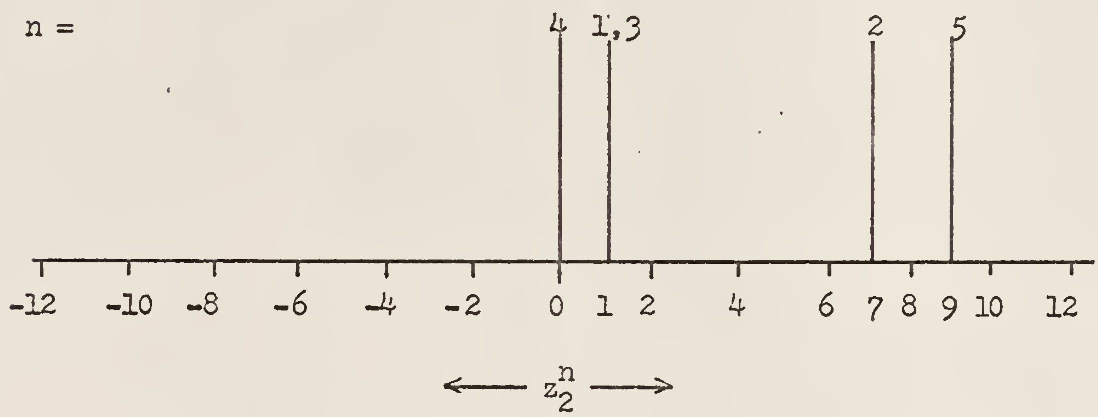

Fig. 5b. Boundary values of adjoint vector z_2^n.

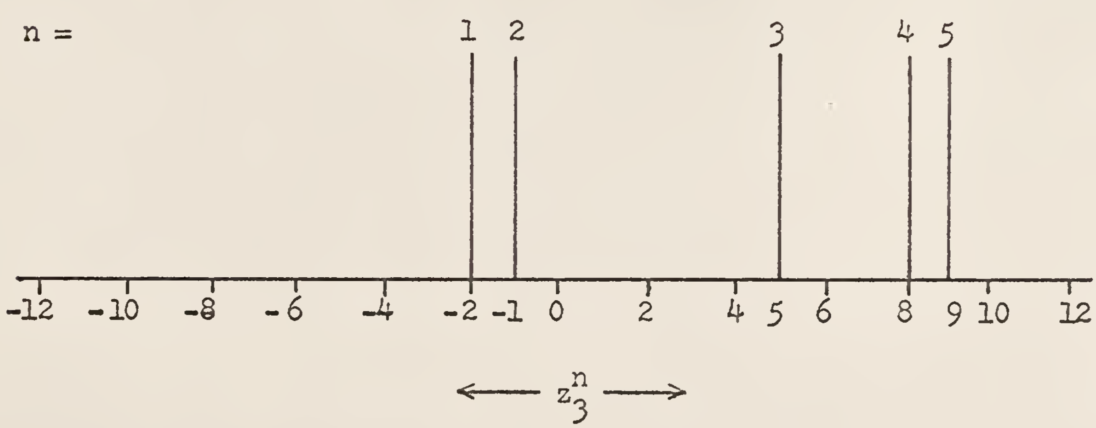

Fig. 5c. Boundary values of adjoint vector z_3^n.

Table 11. Boundary values of z_i^n.

n \ i	1	2	3
1	1	1	- 2
2	4	7	- 1
3	3	1	5
4	6	0	8
5	4	9	9

There is no feasible solution for the conditions $z_1^n = 6$, $z_2^n = 9$ and $z_3^n > -1$.

The feasible solution for conditions $z_1^n = 6$, $z_2^n = 9$ and $z_3^n = -1$ is presented by Table 12.

Table 12. θ_i^n corresponding to the values $z_1^n = 6$, $z_2^n = 9$, and $z_3^n = -1$.

n \ i	1	2	3	4	D^n
1	0	0	0	10	10
2	0	0	(6) $0 \leq \theta_3^2 \leq 20$	(14)	20
3	0	0	10	(0)	10
4	(16) $0 \leq \theta_1^4 \leq 18$	0	(2) $0 \leq \theta_3^4 \leq 18$	(0)	18
5	0	(20) $0 \leq \theta_2^5 \leq 20$	(0) $0 \leq \theta_3^5 \leq 20$	(0)	20
W_i	16	20	18	24	78

The total cost for the above solution = $384.00.

The conditions $z_1^n = 6$, $z_2^n = 9$ and $z_3^n < -1$ give the feasible solution but is not considered as it involves one more undecided control variable.

The feasible solution for conditions $z_1^n = 6$, $z_2^n = 7$ and $z_3^n = 9$ is presented by Table 13.

Table 13. θ_i^n corresponding to the values of $z_1^n = 6$, $z_2^n = 7$, and $z_3^n = 9$.

$n \backslash i$	1	2	3	4	D^n
1	0	0	0	10	10
2	0	(18) $0 \leq \theta_2^2 \leq 20$	0	(2)	20
3	0	0	0	10	10
4	(16) $0 \leq \theta_1^4 \leq 18$	0	0	(2)	18
5	0	(2) $0 \leq \theta_2^5 \leq 20$	(18) $0 \leq \theta_3^5 \leq 20$	(0)	20
W_i	16	20	18	24	78

The total cost for the above solution = \$318.00.

The conditions $z_1^n = 6$, $z_2^n = 7$ and $z_3^n = 5$ or $z_3^n < 9$ give feasible solutions but also increase the number of undecided control variables and hence is not considered.

The feasible solution for conditions $z_1^n = 6$, $z_2^n = 1$ and $z_3^n = 9$ is presented by Table 14.

The total cost for the above solution is \$322.00.

The conditions $z_1^n = 6$, $z_2^n = 1$ and $z_3^n < 9$ are not considered as they involve more undecided control variables.

Table 14. θ_i^n corresponding to the values $z_i^n = 6$, $z_2^n = 1$, and $z_3^n = 9$.

n \ i	1	2	3	4	D^n
1	0	(0) $0 \leq \theta_2^1 \leq 10$	0	(10)	10
2	0	20	0	(0)	20
3	0	(0) $0 \leq \theta_2^3 \leq 10$	0	(10)	10
4	(16) $0 \leq \theta_1^4 \leq 18$	0	0	(2)	18
5	0	(0) $0 \leq \theta_2^5 \leq 20$	(18) $0 \leq \theta_3^5 \leq 20$	(2)	20
W_i	16	20	18	24	78

The feasible solution for the conditions $z_1^n = 4$, $z_2^n = 9$ and $z_3^n = 8$ is presented by Table 15.

Table 15. θ_i^n corresponding to the values $z_1^n = 4$, $z_2^n = 9$, and $z_3^n = 8$.

n \ i	1	2	3	4	D^n
1	0	0	0	10	10
2	(16) $0 \leq \theta_1^2 \leq 20$	0	0	(4)	20
3	0	0	0	10	10
4	(0) $0 \leq \theta_1^4 \leq 18$	0	(18) $0 \leq \theta_3^4 \leq 18$	(0)	18
5	(0) $0 \leq \theta_1^5 \leq 20$	(20) $0 \leq \theta_2^5 \leq 20$	(0) $0 \leq \theta_3^5 \leq 20$	(0)	20
W_i	16	20	18	24	78

The total cost for the above solution is \$332.00.

The conditions $z_1^n = 4$, $z_2^n = 9$ and $z_3^n < 8$ are not considered as they involve more undecided control variables.

The feasible solution for conditions $z_1^n = 4$, $z_2^n = 7$, and $z_3^n = 8$ is presented by Table 16.

Table 16. θ_i^n corresponding to the values $z_1^n = 4$, $z_2^n = 7$, and $z_3^n = 8$.

n \ i	1	2	3	4	D^n
1	0	0	0	10	10
2	$0 \leq \theta_1^2 \leq 20$	$0 \leq \theta_2^2 \leq 20$	0		20
3	0	0	0	10	10
4	$0 \leq \theta_1^4 \leq 18$	0	$0 \leq \theta_3^4 \leq 18$		18
5	$0 \leq \theta_1^5 \leq 20$	$0 \leq \theta_2^5 \leq 20$	$0 \leq \theta_3^5 \leq 20$		20
W_i	16	20	18	24	78

This solution involves too many undecided control variables; therefore, no final solution is obtained here.

Comparing the cost of all feasible solutions and the number of undecided control variables, the solution which gives the least cost and the least undecided control variables is chosen. This is met by the solution given in Table 13.

The next is to try the feasible solution which has the z_i^n in the vicinity of the z_i^n given by Table 13. This new feasible solution may have

one or more undecided control variables than the one given by Table 13. Then the total cost will be compared, and the optimal solution obtained.

In this problem, the feasible solution for conditions $z_1^n = 6$, $z_2^n = 7$ and $z_3^n = 8$ is presented by Table 17. This has one more undecided control variable than the one given by Table 13. The resulting solution from the above is presented by Table 18.

Table 17. θ_i^n corresponding to the values $z_1^n = 6$, $z_2^n = 7$, and $z_3^n = 8$.

n \ i	1	2	3	4	D^n
1	0	0	0	10	10
2	0	$0 \leq \theta_2^2 \leq 20$	0	$20 - \theta_2^2$	20
3	0	0	0	10	10
4	(16) $0 \leq \theta_1^4 \leq 18$	0	$0 \leq \theta_3^4 \leq 18$	$18 - (\theta_1^4 + \theta_3^4)$	18
5	0	$0 \leq \theta_2^5 \leq 20$	$0 \leq \theta_3^5 \leq 18$	$20 - (\theta_2^5 + \theta_3^5)$	20
W_i	θ_1^4 (16)	$\theta_2^2 + \theta_2^5$ (20)	$\theta_3^4 + \theta_3^5$ (18)	(24)	78

The total cost for this solution is $316.00.

The solution given by Table 18 is the optimal solution. This is in contrast to the fact that the feasible solution having least number of undecided control variables usually gives the optimal solution.

This method is still not perfect. There should exist some better methods which may be found in future research work.

Table 18. The optimal solution for z_i^n in the regions of $z_1^n = 6$, $z_2^n = 7$, and $z_3^n = 8$.

n \ i	1	2	3	4	D^n
1	0	0	0	10	10
2	0	(16)	0	(4)	20
3	0	0	0	10	10
4	(16)	0	(2)	(0)	18
5	0	(4)	(16)	(0)	20
W_i	(16)	(20)	(18)	(24)	78

The solution given by Table 18 is the same as given by Simplex Technique for solving such problems [13].

EXAMPLE (4) TWO ORIGINS AND THREE DEMAND POINTS (NON-LINEAR COST FUNCTION)

The non-linear cost function is expressed here by

$$F_i^n\ (\theta_i^n) = a_i^n\ \theta_i^n + b_i^n\ (\theta_i^n)^2$$

where a_i^n, b_i^n are constants. The values of a_i^n, b_i^n with D^n and W_i are shown in Table 19.

Table 19. Transportation costs and requirements for Example (4)

Depots

Demand points: n \ i	1: a_1^n	1: b_1^n	2: a_2^n	2: b_2^n	D^n
1	1.0		3.0		10
2	3.0	0.01	2.1		45
3	3.0		1.0	0.2	20
W_i	30		45		75

The variable part of the Hamiltonian equation for this problem is

$$H_V^n = \sum_{i=1}^{2-1} z_i^n\ \theta_i^n + \sum_{i=1}^{2} F_i^n\ (\theta_i^n)$$

$$= z_1^n\ \theta_1^n + \left\{a_1^n\ \theta_1^n + b_1^n\ (\theta_1^n)^2 + a_2^n\ \theta_2^n + b_2^n\ (\theta_2^n)^2\right\}.$$

Since $\theta_2^n = D^n - \theta_1^n$, then

$$H_V^n = (z_1^n + a_1^n - a_2^n - 2b_2^n\ D^n)\ \theta_1^n + (b_1^n + b_2^n)(\theta_1^n)^2 + a_2^n\ D^n + b_2^n\ (D^n)^2$$

Stage 1:

The variable part of the Hamiltonian equation for the first demand

point (stage) is

$$H_v^1 = (z_1^1 - 2)\, \theta_1^1 + 30$$

This stage is the linear cost function; therefore, it can be treated as shown previously for linear cost function case. Thus

$$\bar{z}_1^1 = 2 = a_2^1 - a_1^1$$

From this three conditions at which H_v^1 may be minimum result:

(a) $H_v^1 = \text{min. at} \quad \theta_1^1 = 0 \quad \text{when } z_1^1 > 2.$

(b) $H_v^1 = \text{min. at } 0 \leq \theta_1^1 \leq 10 \text{ when } z_1^1 = 2.$

(c) $H_v^1 = \text{min. at} \quad \theta_1^1 = 10 \text{ when } z_1^1 < 2.$

Stage 2:

The variable part of the Hamiltonian equation for the second demand point (stage) is

$$H_v^2 = (z_1^2 + .9)\, \theta_1^2 + .01\, (\theta_1^2)^2 + 94.5$$

Taking partial derivative of H_v^2 with respect to θ_1^2 and equating it to zero, the following is obtained:

$$\frac{\partial H_v^2}{\partial \theta_1^2} = 0 = z_1^2 + .9 + .02\, \theta_1^2$$

$$\therefore \theta_1^2 = -45 - 50\, z_1^2$$

when $\theta_1^2 = 0, \qquad z_1^2 = -.9$

and when $\theta_1^2 = 45, \qquad z_1^2 = -1.8$

$$H_v^2 = \text{min., at } \theta_1^2 = 0 \text{ if } z_1^2 \geq -0.9$$

$$\text{and at } \theta_1^2 = 45 \text{ if } z_1^2 \leq -1.8$$

Hence, H_v^2 is minimum at $\theta_1^2 = -45 - 50\, z_1^2$ if $-1.8 \leq z_1^2 \leq -0.9$.

Stage 3:

The variable part of the Hamiltonian equation for the third demand point (stage) is

$$H_v^3 = (z_1^3 - 6)\ \theta_1^3 + 0.2\ (\theta_1^3)^2 + 100$$

Taking partial derivative of H_v^3 with respect to θ_1^3 and equating it to zero results in

$$\frac{\partial H_v^3}{\partial \theta_1^3} = 0 = z_1^3 - 6 + 0.4\ \theta_1^3$$

$$\therefore\ \theta_1^3 = 15 - 2.5\ z_1^3$$

when $\theta_1^3 = 0$, $z_1^3 = 6$

and when $\theta_1^3 = 20$, $z_1^3 = -2$

Hence, H_v^3 is minimum at $\theta_1^3 = 15 - 2.5\ z_1^3$ if $-2 \leq z_1^3 \leq 6$

The conditions for all H_v^n to be minimum are summarized in Table 20.

Table 20. Conditions necessary for H_v^n to be minimum for Example (4).

n	Minima occur at	
	θ_1^n	z_1^n
1	0	> 2
	$0 \leq \theta_1^1 \leq 10$	$= 2$
	10	< 2
2	0	$\geq -.9$
	$-45 - 50\ z_1^2$	$-1.8 \leq z_1^2 \leq -.9$
	45	≤ -1.8
3	0	≥ 6
	$15 - 2.5\ z_1^3$	$-2 \leq z_1^3 \leq 6$
	20	≤ -2

The value of z_1 can now be determined by the condition

$$\sum_{n=1}^{3} \theta_1^n = 30$$

By systematic search for the value of z_1 which satisfies this given condition, the optimal solution will result.

For instance, for the value of z_1 in the region of $-2 < z_1^n < -1.8$, the solution corresponding to this value of z_1 will be

n \ i	1	2	D^n
1	10	0	10
2	45	0	45
3	$15 - 2.5 z_1$	$20 - \theta_1^3$	20
W_i	(30)	(45)	75

This does not satisfy the end-point condition $\sum_{n=1}^{3} \theta_1^n = 30$.
Next, the value of z_1 in the region of $-1.8 < z_1^n < -0.9$.

n \ i	1	2	D^n
1	10	0	10
2	$-45 - 50 z_1$	$45 - \theta_1^2$	45
3	$15 - 2.5 z_1$	$20 - \theta_1^3$	20
W_i	(30)	(45)	75

This gives $-20 - 52.5 z_1 = 30$

$$\therefore z_1 = \frac{50}{-52.5} = -0.9525$$

Here $\sum_{n=1}^{3} \theta_1^n = 30.00625$

Hence, $z_1 = -0.9525$ satisfies the given end-point condition. The result is presented by Table 21. Substituting this value of z_1, then $\theta_1^2 = 2.62$ and $\theta_1^3 = 17.38$. In practical situation, there cannot be a fraction of a unit, so these figures are rounded off to the nearest whole number.

Table 21. θ_i^n corresponding to the value of $z_1 = -0.9525$ for Example (4).

n \ i	1	2	D^n
1	10	0	10
2	(3)	42	45
3	(17)	3	20
W_i	30	45	75

The total cost for the above solution is

$$\sum_{n=1}^{3} \sum_{i=1}^{2} F_i^n (\theta_i^n) = \$163.09$$

This is the only feasible solution. Checking the condition of optimality of the solution given by Table 21 by the perturbation method results in

n \ i	1	2	D^n
1	10	0	10
2	4	41	45
3	16	4	20
W_i	30	45	75

The total cost for the above is

$$\sum_{n=1}^{3} \sum_{i=1}^{2} F_i^n (\theta_i^n) = \$163.46$$

Hence, the optimal solution is that given by Table 21.

EXAMPLE (5). THREE ORIGINS AND THREE DEMAND POINTS
(NON-LINEAR COST FUNCTION)

The problem is represented by Table 22.

Table 22. Transportation costs and requirements for Example (5).

Demand points: n \ Depots: i	1		2		3		D^n
	a_1^n	b_1^n	a_2^n	b_2^n	a_3^n	b_3^n	
1	2.5		2.6		1.0		20
2	3.0	.01	2.7		9.0		60
3	6.0		5.0	.01	6.6		40
W_i	50		30		40		120

The variable part of the Hamiltonian equation for this problem is

$$H_v^n = \sum_{i=1}^{2} z_i^n \theta_i^n + \sum_{i=1}^{3} F_i^n (\theta_i^n)$$

$$= z_1^n \theta_1^n + z_2^n \theta_2^n + \left\{a_1^n \theta_1^n + b_1^n (\theta_1^n)^2\right\} + \left\{a_2^n \theta_2^n + b_2^n (\theta_2^n)^2\right\}$$

$$+ \left\{a_3^n \theta_3^n + b_3^n (\theta_3^n)^2\right\}, \qquad n = 1, 2, 3.$$

Since $\theta_3^n = D^n - \theta_1^n - \theta_2^n$, then

$$H_v^n = (z_1^n + a_1^n - a_3^n - 2b_3^n D^n)\, \theta_1^n + (z_2^n + a_2^n - a_3^n - 2b_3^n D^n)\, \theta_2^n$$

$$+ (b_1^n + b_3^n) (\theta_1^n)^2 + (b_2^n + b_3^n) (\theta_2^n)^2 + a_3^n D^n + b_3^n (D^n)^2$$

$$n = 1, 2, 3.$$

Stage 1:

The variable part of the Hamiltonian equation for the first demand

point (stage) is

$$H_v^1 = (z_1^1 + 1.5)\ \theta_1^1 + (z_2^1 + 1.6)\ \theta_2^1 + 20$$

This stage is the linear cost function and can be treated as previously illustrated. Here,

$$\bar{z}_1^1 = -1.5 = a_3^1 - a_1^1, \text{ and } \bar{z}_2^1 = -1.6 = a_3^1 - a_2^1$$

Following are the nine conditions at which H_v^1 may be minimum:

(a) H_v^1 = min. at $\theta_1^1 = 0$ & $\theta_2^1 = 0$ when $z_1^1 > -1.5$ & $z_2^1 > -1.6$

(b) H_v^1 = min. at $0 \leq \theta_1^1 \leq 20$ & $\theta_2^1 = 0$ when $z_1^1 = -1.5$ & $z_2^1 > -1.6$

(c) H_v^1 = min. at $\theta_1^1 = 20$ & $\theta_2^1 = 0$ when $z_1^1 < -1.5$ & $z_2^1 > -1.6$

(d) H_v^1 = min. at $\theta_1^1 = 0$ & $0 \leq \theta_2^1 \leq 20$ when $z_1^1 > -1.5$ & $z_2^1 = -1.6$

(e) H_v^1 = min. at $0 \leq \theta_1^1 \leq 20$ & $0 \leq \theta_2^1 \leq 20$ when $z_1^1 = -1.5$ & $z_2^1 = -1.6$

(f) H_v^1 = min. at $\theta_1^1 = 20$ & $\theta_2^1 = 0$ when $z_1^1 < -1.5$ & $z_2^1 = -1.6$

(g) H_v^1 = min. at $\theta_1^1 = 0$ & $\theta_2^1 = 20$ when $z_1^1 > -1.5$ & $z_2^1 < -1.6$

(h) H_v^1 = min. at $\theta_1^1 = 0$ & $\theta_2^1 = 20$ when $z_1^1 = -1.5$ & $z_2^1 < -1.6$

(i) H_v^1 = min. at $0 \leq \theta_1^1 \leq 20$ & $0 \leq \theta_2^1 \leq 20$ when $z_1^1 < -1.5$ & $z_2^1 < -1.6$

Stage 2:

The variable part of the Hamiltonian equation for the second demand point (stage) is

$$H_v^2 = (z_1^2 - 6)\ \theta_1^2 + (z_2^2 - 6.9)\ \theta_2^2 + 0.01\ (\theta_1^2)^2 + 540$$

Taking partial derivative of H_v^2 with respect to θ_1^2 and equating it to zero

results in

$$\frac{\partial H_v^2}{\partial \theta_1^2} = z_1^2 - 6 + 0.02\ \theta_1^2$$

$\theta_1^2 = 300 - 50\ z_1^2$

when $\theta_1^2 = 0$, $z_1^2 = 6$

and when $\theta_1^2 = 60$, $z_1^2 = 4.8$.

Hence, H_v^2 is minimum at $\theta_1^2 = 300 - 50\ z_1^2$ for $4.8 \leq z_1^2 \leq 6$.

Following are the nine conditions at which H_v^2 may be minimum.

(a) $H_v^2 = \text{min. at } \theta_1^2 = 0$ & $\theta_2^2 = 0$ when $z_1^2 \geq 6$ & $z_2^2 > 6.9$

(b) $H_v^2 = \text{min. at } \theta_1^2 = 0$ & $0 \leq \theta_2^2 \leq 60$ when $z_1^2 \geq 6$ & $z_2^2 = 6.9$

(c) $H_v^2 = \text{min. at } \theta_1^2 = 0$ & $\theta_2^2 = 60$ when $z_1^2 \geq 6$ & $z_2^2 < 6.9$

(d) $H_v^2 = \text{min. at } \theta_1^2 = 300 - 50\ z_1^2$ & $\theta_2^2 = 0$ when $4.8 \leq z_1^2 < 6$ & $z_2^2 > 6.9$

(e) $H_v^2 = \text{min. at } \theta_1^2 = 300 - 50\ z_1^2$ & $0 \leq \theta_2^2 \leq 60$ when $4.8 \leq z_1^2 < 6$ & $z_2^2 = 6.9$

(f) $H_v^2 = \text{min. at } \theta_1^2 = 300 - 50\ z_1^2$ & $0 \leq \theta_2^2 \leq 60$ when $4.8 \leq z_1^2 < 6$ & $z_2^2 < 6.9$

(g) $H_v^2 = \text{min. at } \theta_1^2 = 60$ & $\theta_2^2 = 0$ when $z_1^2 < 4.8$ & $z_2^2 > 6.9$

(h) $H_v^2 = \text{min. at } \theta_1^2 = 60$ & $\theta_2^2 = 0$ when $z_1^2 < 4.8$ & $z_2^2 = 6.9$

(i) $H_v^2 = \text{min. at } 0 \leq \theta_1^2 \leq 60$ & $0 \leq \theta_2^2 \leq 60$ when $z_1^2 < 4.8$ & $z_2^2 < 6.9$

Stage 3:

The variable part of the Hamiltonian equation for the third demand point (stage) is

$$H_v^3 = (z_1^3 - 0.6)\,\theta_1^3 + (z_2^3 - 1.6)\,\theta_2^3 + 0.01\,(\theta_2^3)^2 + 264$$

Taking partial derivative of H_v^3 with respect to θ_2^3 and equating it to zero results in

$$\frac{\partial H_v^3}{\partial \theta_2^3} = 0 = z_2^3 - 1.6 + 0.02\,\theta_2^3$$

$$\therefore\ \theta_2^3 = 80 - 50\,z_2^3$$

when $\theta_2^3 = 0$, $z_2^3 = 1.6$

and when $\theta_2^3 = 40$, $z_2^3 = 0.8$

Hence, H_v^3 is minimum at $\theta_2^3 = 80 - 50\,z_2^3$ for $0.8 \le z_2^3 \le 1.6$

Following are the nine conditions at which H_v^3 may be minimum.

(a) H_v^3 = min. at $\theta_1^3 = 0$ & $\theta_2^3 = 0$ when $z_1^3 > .6$ & $z_2^3 \geq 1.6$

(b) H_v^3 = min. at $\theta_1^3 = 0$ & $\theta_2^3 = 80 - 50\,z_2^3$ when $z_1^3 > .6$ & $.8 \le z_2^3 < 1.6$

(c) H_v^3 = min. at $\theta_1^3 = 0$ & $\theta_2^3 = 40$ when $z_1^3 > .6$ & $z_2^3 < .8$

(d) H_v^3 = min. at $0 \le \theta_1^3 \le 40$ & $\theta_2^3 = 0$ when $z_1^3 = .6$ & $z_2^3 \geq 1.6$

(e) H_v^3 = min. at $0 \le \theta_1^3 \le 40$ & $\theta_2^3 = 80 - 50\,z_2^3$ when $z_1^3 = .6$ & $.8 \le z_2^3 < 1.6$

(f) H_v^3 = min. at $\theta_1^3 = 0$ & $\theta_2^3 = 40$ when $z_1^3 = .6$ & $z_2^3 < 0.8$

(g) H_v^3 = min. at $\theta_1^3 = 40$ & $\theta_2^3 = 0$ when $z_1^3 < .6$ & $z_2^3 \geq 1.6$

(h) H_v^3 = min. at $0 \le \theta_1^3 \le 40$ & $\theta_2^3 = 80 - 50\,z_2^3$ when $z_1^3 < .6$ & $.8 \le z_2^3 < 1.6$

(i) H_v^3 = min. at $0 \le \theta_1^3 \le 40$ & $0 \le \theta_2^3 \le 40$ when $z_1^3 < .6$ & $z_2^3 < 0.8$

The conditions for all H_v^n to be minimum are summarized in Table 23.

Table 23. Conditions necessary for H_v^n to be minimum for Example (5).

n	Minima occur at			
	θ_1^n	θ_2^n	z_1^n	z_2^n
1	0	0	> -1.5	> -1.6
	$0 \leq \theta_1^1 \leq 20$	0	$= -1.5$	> -1.6
	20	0	< -1.5	> -1.6
	0	$0 \leq \theta_2^1 \leq 20$	> -1.5	$= -1.6$
	$0 \leq \theta_1^1 \leq 20$	$0 \leq \theta_2^1 \leq 20$	$= -1.5$	$= -1.6$
	20	0	< -1.5	$= -1.6$
	0	20	> -1.5	< -1.6
	0	20	$= -1.5$	< -1.6
	$0 \leq \theta_1^1 \leq 20$	$0 \leq \theta_2^1 \leq 20$	< -1.5	< -1.6
2	0	0	≥ 6	> 6.9
	0	$0 \leq \theta_2^2 \leq 60$	≥ 6	$= 6.9$
	0	60	≥ 6	< 6.9
	$300 - 50\, z_1^2$	0	$4.8 \leq z_1^2 < 6$	> 6.9
	$300 - 50\, z_1^2$	$0 \leq \theta_2^2 \leq 60$	$4.8 \leq z_1^2 < 6$	$= 6.9$
	$300 - 50\, z_1^2$	$0 \leq \theta_2^2 \leq 60$	$4.8 \leq z_1^2 < 6$	< 6.9
	60	0	< 4.8	> 6.9
	60	0	< 4.8	$= 6.9$
	$0 \leq \theta \leq 60$	$0 \leq \theta \leq 60$	< 4.8	< 6.9

Table 23. Conditions necessary for H_v^n to be minimum for Example (5) (Continued)

n	θ_1^n	θ_2^n	z_1^n	z_2^n
	0	0	$> .6$	≥ 1.6
	0	$80 - 50\ z_2^3$	$> .6$	$.8 \leq z_2^3 < 1.6$
	0	40	$> .6$	< 0.8
	$0 \leq \theta_1^3 \leq 40$	0	$= .6$	≥ 1.6
3	$0 \leq \theta_1^3 \leq 40$	$80 - 50\ z_2^3$	$= .6$	$.8 \leq z_2^3 < 1.6$
	0	40	$= .6$	< 0.8
	40	0	$< .6$	≥ 1.6
	$0 \leq \theta_1^3 \leq 40$	$80 - 50\ z_2^3$	$< .6$	$.8 \leq z_2^3 < 1.6$
	$0 \leq \theta_1^3 \leq 40$	$0 \leq \theta_2^3 \leq 40$	$< .6$	< 0.8

By systematically searching each combination of the interior and/or the boundary values of z_1^n and z_2^n for feasible solutions, cases which do not satisfy the constraints are eliminated.

A possible feasible solution corresponding to the values of z_1^n in the region of $4.8 < z_1^n < 6$ and z_2^n in the region of $0.8 < z_2^n < 1.6$ which satisfies the constraints is presented in Table 24.

Here two conditions are obtained:

(i) $80 - 50\ z_2^n = 30$ which gives $z_2^n = 1.$

(ii) $90 - 50\ z_2^n = 30$ which gives $z_2^n = 1.2.$

It is not advisable to consider condition (i) given above since it gives higher total cost. The feasible solution given by condition (ii) is also presented in Table 24. The total cost for this solution is \$452.00.

Table 24. θ_i^n corresponding to the values of z_1^n and z_2^n in the regions of $4.8 < z_1^n < .6$, and $0.8 < z_2^n < 1.6$.

n \ i	1	2	3	D^n
1	0	0	20	20
2	50	(10) $0 \leq \theta_2^2 \leq 60$	(0)	60
3	0	(20) $80 - 50\, z_2^n$	(20)	40
W_i	(50)	(30)	(40)	120

Another possible feasible solution corresponding to the values $z_1^n = 0.6$ and $z_2^n = 1.6$ is presented by Table 25.

Table 25. θ_i^n corresponding to the values $z_1^n = 0.6$ and $z_2^n = 1.6$.

n \ i	1	2	3	D^n
1	0	0	20	20
2	$0 \leq \theta_1^2 \leq 60$	$0 \leq \theta_2^2 \leq 60$	$60 - (\theta_1^2 + \theta_2^2)$	60
3	$0 \leq \theta_1^3 \leq 40$	0	$40 - \theta_1^3$	40
W_i	(50)	(30)	(40)	120

A feasible solution resulting from Table 25 is given by Table 26. The solution of Table 26 is obtained as follows:

As there is no other choice, $\theta_2^2 = 30$ has to satisfy the end condition $W_2 = 30$. Comparing the cost functions for deciding among the control variables θ_1^2, θ_3^2, θ_1^3 and θ_3^3, it is advantageous to assign maximum value

to the control variable θ_1^2 by comparing the costs given by Table 22. Then $\theta_1^2 = D^2 - \theta_2^2$ which is $\theta_1^2 = 60 - 30 = 30$ units can be assigned. After θ_1^2, the rest of the solution is obtained by meeting the end-point constraints.

Table 26. The optimal solution for $z_1^n = 0.6$ and $z_2^n = 1.6$.

n \ i	1	2	3	D^n
1	0	0	20	20
2	(30)	(30)	(0)	60
3	(20)	0	(20)	40
W_i	50	30	40	120

The total cost for the solution given in Table 26 is

$$\sum_{n=1}^{3} \sum_{i=1}^{3} F_i^n (\theta_i^n) = \$434.00.$$

Comparing the feasible solutions in Tables 24 and 26, Table 26 is the optimum solution.

Checking by perturbation the condition of optimality, the solution is:

n \ i	1	2	3	D^n
1	0	0	20	20
2	31	29	0	60
3	19	1	20	40
W_i	50	30	40	120

The total cost for the above is

$$\sum_{n=1}^{3} \sum_{i=1}^{3} F_i^n (\theta_i^n) = \$434.52.$$

Checking by perturbation the condition of optimality, the solution is:

$n \backslash i$	1	2	3	D^n
1	0	0	20	20
2	29	30	1	60
3	31	0	19	40
W_i	50	30	40	120

The total cost for the above solution is

$$\sum_{n=1}^{3} \sum_{i=1}^{3} F_i^n \, (\theta_i^n) = \$438.81.$$

Hence, the optimal cost of \$434.00 as given by Table 26.

EXAMPLE (6). TWO ORIGINS AND THREE DEMAND POINTS
(NON-LINEAR COST FUNCTION WITH SET UP COST)

The non-linear cost function for this problem is defined as

$$F_i^n\ (\theta_i^n) = a_i^n\ \theta_i^n + b_i^n\ (\theta_i^n)\ \ + C_i^n \left\lfloor \theta_i^n \right\rfloor$$

where a_i^n and b_i^n are constants and $C_i^n \left\lfloor \theta_i^n \right\rfloor$ is called a "set-up" cost or "fixed charge." It is equal to zero if $\theta_i^n = 0$ and is equal to a positive constant C_i^n if $\theta_i^n > 0$.

The problem is represented by Table 27.

Table 27. Transportation costs and requirements for Example (6).

Depots

Demand points: n \ i	1: a_1^n	1: b_1^n	1: c_1^n	2: a_2^n	2: b_2^n	2: c_2^n	D^n
1	2.5			2.6			5
2	6.0			5.0	.01		20
3	5.0	- .01	10	3.0			15
W_i	25			15			40

The variable part of the Hamiltonian equation for this problem is

$$H_v^n = \sum_{i=1}^{2-1} z_i^n\ \theta_i^n + \sum_{i=1}^{2} F_i^n\ (\theta_i^n)$$

$$= z_1^n\ \theta_1^n + \left\{a_1^n\ \theta_1^n + b_1^n\ (\theta_1^n)^2 + C_1^n \left\lfloor \theta^n \right\rfloor\right\} + \left\{a_2^n\ \theta_2^n + b_2^n\ (\theta_2^n)^2 + C_2^n \left\lfloor \theta_2^n \right\rfloor\right\}, \qquad n = 1, 2, 3.$$

Since $\theta_2^n = D^n - \theta_1^n$, then

$$H_v^n = (z_1^n + a_1^n - a_2^n - 2b_2^n D^n)\ \theta_1^n + (b_1^n + b_2^n)\ (\theta_1^n)^2 + C_1^n \lfloor \theta_1^n \rfloor$$
$$+ C_2^n \lfloor D^n - \theta_1^n \rfloor + a_2^n D^n + b_2^n (D^n)^2.$$

Stage 1:

The variable part of the Hamiltonian equation for the first demand point (stage) is

$$H_v^1 = (z_1^1 - 0.1)\ \theta_1^1 + 13.$$

This stage is linear cost function and therefore can be treated in the way shown previously. Thus

$$\bar{z}_1^1 = 0.1 = a_2^1 - a_1^1.$$

From this three conditions at which H_v^1 may be minimum result:

(a) $H_v^1 = \text{min. at} \quad \theta_1^1 = 0$ when $z_1^1 > 0.1$.

(b) $H_v^1 = \text{min. at } 0 \leq \theta_1^1 \leq 5$ when $z_1^1 = 0.1$.

(c) $H_v^1 = \text{min. at} \quad \theta_1^1 = 5$ when $z_1^1 < 0.1$

Stage 2:

The variable part of the Hamiltonian equation for the second demand point (stage) is

$$H_v^2 = (z_1^2 + 0.6)\ \theta_1^2 + 0.01\ (\theta_1^2)^2 + 104.$$

Taking partial derivative of H_v^2 with respect to θ_1^2 and equating it to zero, the following is obtained:

$$\frac{\partial H_v^2}{\partial \theta_1^2} = 0 = z_1^2 + 0.6 + 0.02\ \theta_1^2.$$

$$\therefore\ \theta_1^2 = -30 - 50\, z_1^2$$

when $\theta_1^2 = 0$, $\quad z_1^2 = -0.6$

when $\theta_1^2 = 20$, $\quad z_1^2 = -1$

$$H_v^2 = \text{min., at } \theta_1^2 = 0 \text{ if } z_1^2 \geq -0.6$$

$$\text{and at } \theta_1^2 = 20 \text{ if } z_1^2 \leq -1.$$

Hence, H_v^2 is minimum at $0 \leq \theta_1^2 \leq 20$ if $-1 \leq z_1^2 \leq -0.6$.

Stage 3:

The variable part of the Hamiltonian equation for the third demand point (stage) is

$$H_v^3 = (z_1^3 + 2)\, \theta_1^3 - 0.01\, (\theta_1^3)^2 + 10 \left\lfloor \theta_1^3 \right\rfloor + 45.$$

Here, when $\theta_1^3 = 0$, $H_v^3 = 45$

and when $\theta_1^3 = 15$, then

$$H_v^3 = 45 = (z_1^3 + 2)\, 15 - 0.01\, (15)^2 + 10 + 45$$

$$z_1^3 = -\frac{151}{60}\ .$$

Following are the three conditions at which H_v^3 may be minimum

(a) $H_v^3 = \text{min. at} \quad \theta_1^3 = 0 \quad \text{when } z_1^3 > -\frac{151}{60}$

(b) $H_v^3 = \text{min. at } 0 \leq \theta_1^3 \leq 15 \quad \text{when } z_1^3 = -\frac{151}{60}$

(c) $H_v^3 = \text{min. at} \quad \theta_1^3 = 15 \quad \text{when } z_1^3 < -\frac{151}{60}$

The conditions for all H_v^3 to be minimum are summarized in Table 28.

By systematic search for the value of z_1^n which satisfies the given condition $\sum_{n=1}^{3} \theta_1^n = 25$, the optimal solution will be obtained.

Table 28. Conditions necessary for H_V^n to be minimum for Example (6).

n	Minima occur at θ_1^n	z_1^n
1	0	> 0.1
	$0 \leq \theta_1^1 \leq 5$	$= 0.1$
	5	< 0.1
2	0	≥ 0.6
	$-30 - 50\, z_1^n$	$-1 \leq z_1^n < 0.6$
	20	< -1
3	0	$> -\frac{151}{60}$
	$0 \leq \theta_1^3 \leq 15$	$= -\frac{151}{60}$
	15	$< -\frac{151}{60}$

The region $-0.6 < z_1^n < 0.1$ does not give the feasible solution. Considering the region $-1 < z_1^n < -0.6$, the feasible solution corresponding to this value of z_1^n is given by Table 29.

Table 29. θ_i^n corresponding to the value of z_1 in region of $-1 \leq z_1 \leq -0.6$

n \ i	1	2	D^n
1	5	0	5
2	$-30 - 50\, z_1$	$20 - \theta_1^2$	20
3	0	15	15
W_i	$5 + \theta_1^2$ (25)	$35 - \theta_1^2$ (15)	40

This satisfies the given end-point condition which gives

$$-25 - 50\, z_1 = 25$$

$$\therefore z_1 = -1.$$

The optimal solution corresponding to the value of $z_1 = -1$ is given by Table 30.

Table 30. The optimal solution for $z_1 = -1$.

$n \backslash i$	1	2	D^n
1	5	0	5
2	20	0	20
3	0	15	15
W_i	(25)	(15)	40

The total cost for the solution given by Table 30 is

$$\sum_{n=1}^{3} \sum_{i=1}^{2} F_i^n\, (\theta_i^n) = \$177.50.$$

Checking the condition of optimality of the solution given by Table 30 by the perturbation method results in

$n \backslash i$	1	2	D^n
1	5	0	5
2	19	1	20
3	1	14	15
W_i	(25)	(15)	40

The total cost for the above is

$$\sum_{n=1}^{3} \sum_{i=1}^{2} F_i^n\, (\theta_i^n) = \$183.49.$$

Hence, the optimal solution is that given by Table 30.

EXAMPLE (7). THREE ORIGINS AND FOUR DEMAND POINTS (NON-LINEAR COST FUNCTION WITH SET UP COST)

The problem is represented by Table 31.

Table 31. Transportation costs and requirements for Example (7).

	Depots									
i / n	1			2			3			D^n
Demand points	a_1^n	b_1^n	c_1^n	a_2^n	b_2^n	c_2^n	a_3^n	b_3^n	c_3^n	
1	1.0			3.1		2	7.0			25
2	2.0		1	4.1			3.0			40
3	5.0	- .01	10	3.0			2.0		5	30
4	3.0			1.0	0.2	5	4.0			35
W_i		40			30			60		130

The variable part of the Hamiltonian equation for this problem is

$$H_V^n = \sum_{i=1}^{3-1} z_i^n \theta_i^n + \sum_{i=1}^{3} F_i^n (\theta_i^n), \qquad n = 1, 2, 3, 4.$$

$$= z_1^n \theta_1^n + z_2^n \theta_2^n + \left\{a_1^n \theta_1^n + b_1^n (\theta_1^n)^2 + C_1^n \lfloor \theta^n \rfloor\right\}$$

$$+ \left\{a_2^n \theta_2^n + b_2^n (\theta_2^n)^2 + C_2^n \lfloor \theta_2^n \rfloor\right\} + \left\{a_3^n \theta_3^n + b_3^n (\theta_3^n)^2 + C_3^n \lfloor \theta_3^n \rfloor\right\}$$

Since $\theta_3^n = D^n - \theta_1^n - \theta_2^n$, then

$$H_V^n = (z_1^n + a_1^n - a_3^n - 2b_3^n D^n)\, \theta_1^n + (z_2^n + a_2^n - a_3^n - 2b_3^n D^n)\, \theta_2^n$$

$$+ (b_1^n + b_3^n)(\theta_1^n)^2 + (b_2^n + b_3^n)(\theta_2^n)^2 + b_3^n (D^n)^2$$

$$+ 2b_3^n \theta_1^n \theta_2^n + C_1^n \lfloor \theta_1^n \rfloor + C_2^n \left[\theta_2^n\right] + C_3^n \lfloor D^n - \theta_1^n - \theta_2^n \rfloor + a_3^n D^n$$

Stage 1:

The variable part of the Hamiltonian equation for the first demand point (stage) is

$$H_v^1 = (z_1^1 - 6)\ \theta_1^1 + (z_2^1 - 3.9)\ \theta_2^1 + 2\ \lfloor \theta_2^1 \rfloor + 175$$

when $\theta_1^1 = 0$ and $\theta_2^1 = 0$, $H_v^1 = 175$

$$\bar{z}_1^1 = 6$$

When $z_1^1 > 6$, $\theta_1^1 = 0$, and for $\theta_2^1 = 25$ in the following equation

$$H_v^1 = (z_2^1 - 3.9)\ \theta_2^1 + 2\ \lfloor \theta_2^1 \rfloor + 175 = 175$$

gives $\bar{z}_2^1 = 3.82$

To see the effects of different values of z_2^1 on the Hamiltonian function of H_v^1 when $\theta_1^1 = 0$ and θ_2^1 varying from 0 to 25, a computer program is written for IBM 1620 computer. The results thus obtained are presented graphically (see Fig. 6).

Following are the nine conditions at which H_v^1 may be minimum:

(a) H_v^1 = min. at $\theta_1^1 = 0$ & $\theta_2^1 = 0$ when $z_1^1 > 6$ & $z_2^1 > 3.82$

(b) H_v^1 = min. at $\theta_1^1 = 0$ & $\theta_2^1 = 25$ when $z_1^1 > 6$ & $z_2^1 = 3.82$

(c) H_v^1 = min. at $\theta_1^1 = 0$ & $\theta_2^1 = 25$ when $z_1^1 > 6$ & $z_2^1 < 3.82$

(d) H_v^1 = min. at $0 \leq \theta_1^1 \leq 25$ & $\theta_2^1 = 0$ when $z_1^1 = 6$ & $z_2^1 > 3.82$

(e) H_v^1 = min. at $0 \leq \theta_1^1 \leq 25$ & $\theta_2^1 = 0$ or 25 when $z_1^1 = 6$ & $z_2^1 = 3.82$

(f) H_v^1 = min. at $0 \leq \theta_1^1 \leq 25$ & $0 \leq \theta_2^1 \leq 25$ when $z_1^1 = 6$ & $z_2^1 < 3.82$

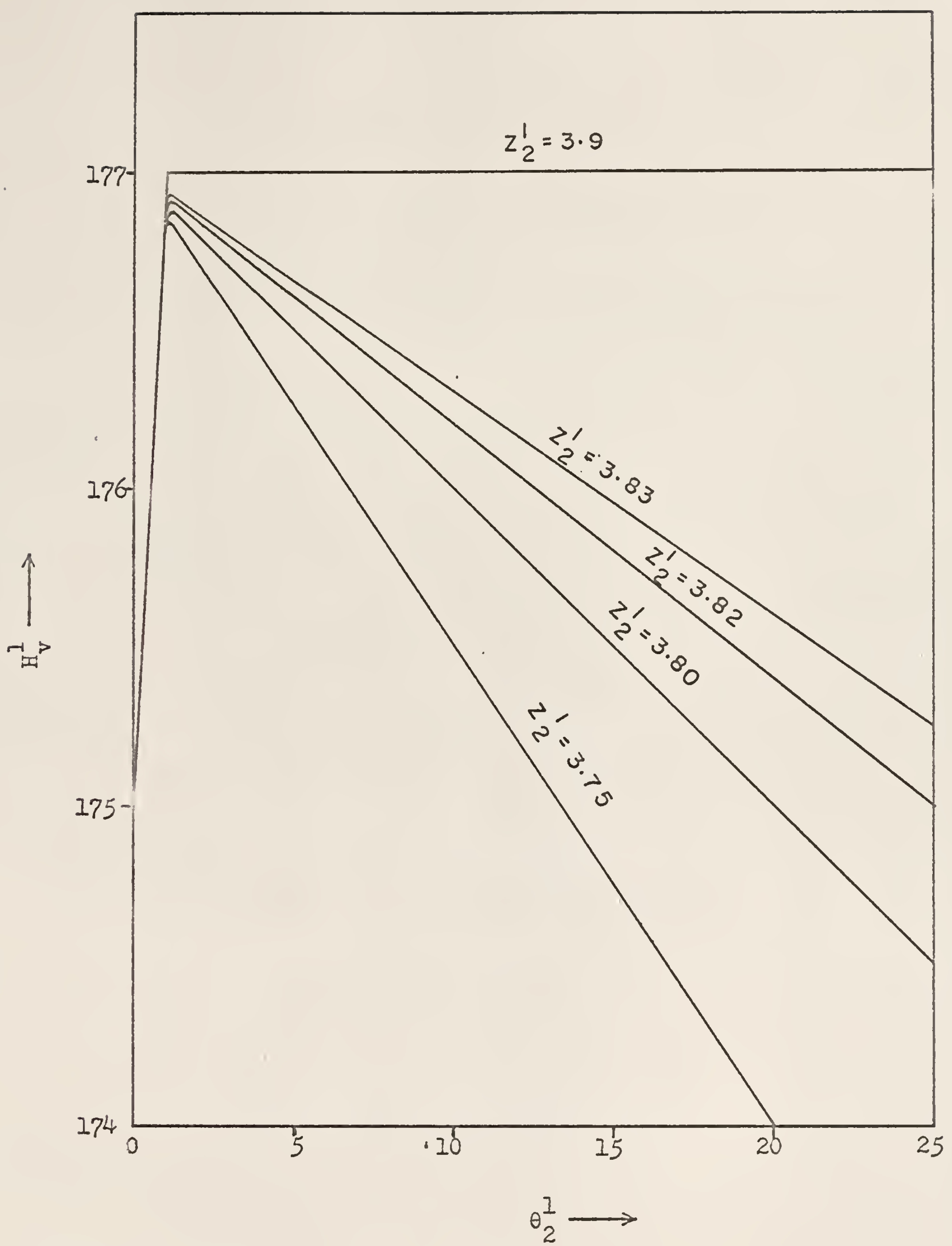

Fig. 6. Effects of z_2^1 on H_v^1 when $\theta_1^1 = 0$, and θ_2^1 varies from 0 to 25.

(g) $H_v^1 = \text{min. at} \quad \theta_1^1 = 25 \ \& \quad \theta_2^1 = 0 \quad \text{when } z_1^1 < 6 \ \& \ z_2^1 > 3.82$

(h) $H_v^1 = \text{min. at} \quad \theta_1^1 = 25 \ \& \quad \theta_2^1 = 0 \quad \text{when } z_1^1 < 6 \ \& \ z_2^1 = 3.82$

(i) $H_v^1 = \text{min. at } 0 \leq \theta_1^1 \leq 25 \ \& \ 0 \leq \theta_2^1 \leq 25 \quad \text{when } z_1^1 < 6 \ \& \ z_2^1 < 3.82$

Stage 2:

The variable part of the Hamiltonian equation for the second demand point (stage) is

$$H_v^2 = (z_1^2 - 1)\ \theta_1^2 + (a_2^2 + 1.1)\ \theta_2^2 + 1 \left\lfloor \theta_1^2 \right\rfloor + 120$$

when $\theta_1^2 = 0$ and $\theta_2^2 = 0$, $H_v^2 = 120$

when $\theta_2^2 = 0$, $\bar{z}_2^2 = -1.1$

$$H_v^2 = 120 = (z_1^2 - 1)\ \theta_1^2 + 1 \left\lfloor \theta_1^2 \right\rfloor + 120$$

$$\text{at } \theta_1^2 = 40$$

$$\bar{z}_1^2 = 0.975$$

Following are the nine conditions at which H_v^2 may be minimum:

(a) $H_v^2 = \text{min. at} \quad \theta_1^2 = 0 \quad \& \quad \theta_2^2 = 0 \quad \text{when } z_1^2 > .975 \ \& \ z_2^2 > -1.1$

(b) $H_v^2 = \text{min. at} \quad \theta_1^2 = 0 \quad \& \ 0 \leq \theta_2^2 \leq 40 \quad \text{when } z_1^2 > .975 \ \& \ z_2^2 = -1.1$

(c) $H_v^2 = \text{min. at} \quad \theta_1^2 = 0 \quad \& \quad \theta_2^2 = 40 \quad \text{when } z_1^2 > .975 \ \& \ z_2^2 < -1.1$

(d) $H_v^2 = \text{min. at } \theta_1^2 = 0 \text{ or } 40 \ \& \quad \theta_2^2 = 0 \quad \text{when } z_1^2 = .975 \ \& \ z_2^2 > -1.1$

(e) $H_v^2 = \text{min. at } \theta_1^2 = 0 \text{ or } 40 \ \& \ 0 \leq \theta_2^2 \leq 40 \quad \text{when } z_1^2 = .975 \ \& \ z_2^2 = -1.1$

(f) $H_v^2 = \text{min. at} \quad \theta_1^2 = 0 \quad \& \quad \theta_2^2 = 40 \quad \text{when } z_1^2 = .975 \ \& \ z_2^2 < -1.1$

(g) $H_v^2 = \text{min. at} \quad \theta_1^2 = 40 \quad \& \quad \theta_2^2 = 0 \quad \text{when } z_1^2 < .975 \;\&\; z_2^2 > -1.1$

(h) $H_v^2 = \text{min. at } 0 \leq \theta_1^2 \leq 40 \quad \& \; 0 \leq \theta_2^2 \leq 40 \quad \text{when } z_1^2 < .975 \;\&\; z_2^2 = -1.1$

(i) $H_v^2 = \text{min. at } 0 \leq \theta_1^2 \leq 40 \quad \& \; 0 \leq \theta_2^2 \leq 40 \quad \text{when } z_1^2 < .975 \;\&\; z_2^2 < -1.1$

Stage 3:

The variable part of the Hamiltonian equation for the third demand point (stage) is

$$H_v^3 = (z_1^3 + 3)\,\theta_1^3 + (z_2^3 + 1)\,\theta_2^3 - 0.01\,(\theta_1^3)^2 + 10\,\lfloor \theta_1^3 \rfloor$$

$$+ 5\,\lfloor 30 - \theta_1^3 - \theta_2^3 \rfloor + 60$$

when $\theta_1^3 = 0$ and $\theta_2^3 = 0$, then

$$H_v^3 = 5 + 60 = 65$$

when $\theta_1^3 = 30$, and $\theta_2^3 = 0$, so

$$H_v^3 = 65 = (z_1^3 + 3)\,\theta_1^3 - 0.01\,(\theta_1^3)^2 + 10\,\lfloor \theta_1^3 \rfloor + 60$$

$$= (z_1^3 + 3)\,30 - 0.01\,(30)^2 + 10 + 60$$

$$\therefore \bar{z}_1^3 = -\frac{86}{30} = -\frac{43}{15}$$

Similarly, when $\theta_1^3 = 0$, and $\theta_2^3 = 30$, then

$$H_v^3 = 65 = (z_2^3 + 1)\,\theta_2^3 + 60$$

$$\therefore \bar{z}_2^3 = -\frac{25}{30} = -\frac{5}{6}$$

Following are the nine conditions at which H_v^3 may be minimum:

(a) H_v^3 = min. at $\theta_1^3 = 0$ & $\theta_2^3 = 0$ when $z_1^3 > -\frac{43}{15}$ & $z_2^3 > -\frac{5}{6}$

(b) H^3 = min. at $\theta_1^3 = 0$ & $\theta_2^3 = 0$ or 30 when $z_1^3 > -\frac{43}{15}$ & $z_2^3 = -\frac{5}{6}$

(c) H_v^3 = min. at $\theta_1^3 = 0$ & $\theta_2^3 = 30$ when $z_1^3 > -\frac{43}{15}$ & $z_2^3 < -\frac{5}{6}$

(d) H_v^3 = min. at $\theta_1^3 = 0$ or 30 & $\theta_2^3 = 0$ when $z_1^3 = -\frac{43}{15}$ & $z_2^3 > -\frac{5}{6}$

(e) H_v^3 = min. at $\theta_1^3 = 0$ or 30 & $\theta_2^3 = 0$ or 30 when $z_1^3 = -\frac{43}{15}$ & $z_2^3 = -\frac{5}{6}$

(f) H_v^3 = min. at $\theta_1^3 = 0$ & $\theta_2^3 = 30$ when $z_1^3 = -\frac{43}{15}$ & $z_2^3 < -\frac{5}{6}$

(g) H_v^3 = min. at $\theta_1^3 = 30$ & $\theta_2^3 = 0$ when $z_1^3 < -\frac{43}{15}$ & $z_2^3 > -\frac{5}{6}$

(h) H_v^3 = min. at $\theta_1^3 = 30$ & $\theta_2^3 = 0$ when $z_1^3 < -\frac{43}{15}$ & $z_2^3 = -\frac{5}{6}$

(i) H_v^3 = min. at $0 \leq \theta_1^3 \leq 30$ & $0 \leq \theta_2^3 \leq 30$ when $z_1^3 < -\frac{43}{15}$ & $z_2^3 < -\frac{5}{6}$

Stage 4:

The variable part of the Hamiltonian equation for the fourth demand point (stage) is

$$H_v^4 = (z_1^4 - 1)\,\theta_1^4 + (z_2^4 - 3)\,\theta_2^4 + 0.2\,(\theta_2^4)^2 + 5\,\lfloor \theta_2^4 \rfloor + 140$$

Taking partial derivative of H_v^4 with respect to θ_2^4, and equating it to zero gives

$$\frac{\partial H_v^4}{\partial \theta_2^4} = 0 = z_2^4 - 3 + 0.4\,\theta_2^4$$

$$\therefore \theta_2^4 = 7.5 - 2.5\,z_2^4 \quad \text{or} \quad z_2^4 = 3 - 0.4\,\theta_2^4$$

Considering $\theta_1^4 = 0$ and

$$H_v^4 = (z_2^4 - 3)\,\theta_2^4 + 0.2\,(\theta_2^4)^2 + 5\,\lfloor \theta_2^4 \rfloor + 140$$

Taking partial derivative of H_v^4 with respect to θ_2^4 and equating it to zero gives

$$\frac{\partial H_v^4}{\partial \theta_2^4} = 0 = z_2^4 - 3 + 0.4\ \theta_2^4$$

$$\therefore\ \theta_2^4 = 7.5 - 2.5\ z_2^4$$

or $$z_2^4 = 3 - 0.4\ \theta_2^4$$

For $\theta_2^4 = 35$

$$z_2^4 = 3 - 0.4\ (35) = -\ 11$$

For $\theta_2^4 = 0$,

$$z_2^4 = 3$$

Referring to Fig. 7, some intermediate point for the condition $z_2^4 = 3 - 0.4\ \theta_2^4$ at which $H_v^4 = 140$ is found. For this, the value of θ_2^4 follows:

$$H^4 = 140 = (3 - 0.4\ \theta_2^4 - 3)\ \theta_2^4 + 0.2\ (\theta_2^4)^2 + 5 \left\lfloor \theta_2^4 \right\rfloor + 140$$

$$-\ 5 = -\ 0.4\ (\theta_2^4)^2 + 0.2\ (\theta_2^4)^2$$

$$= -\ 0.2\ (\theta_2^4)^2$$

$$\therefore\ (\theta_2^4)^2 = 25$$

and $$\theta_2^4 = 5$$

Hence, the value of z_2^4 corresponding to the value of $\theta_2^4 = 5$ is

$$z_2^4 = 3 - 0.4\ (5) = 3 - 2 = 1$$

H_v^4 = minimum when z_2^4 is in the region $-\ 11 \leq z_2^4 \leq 1$.

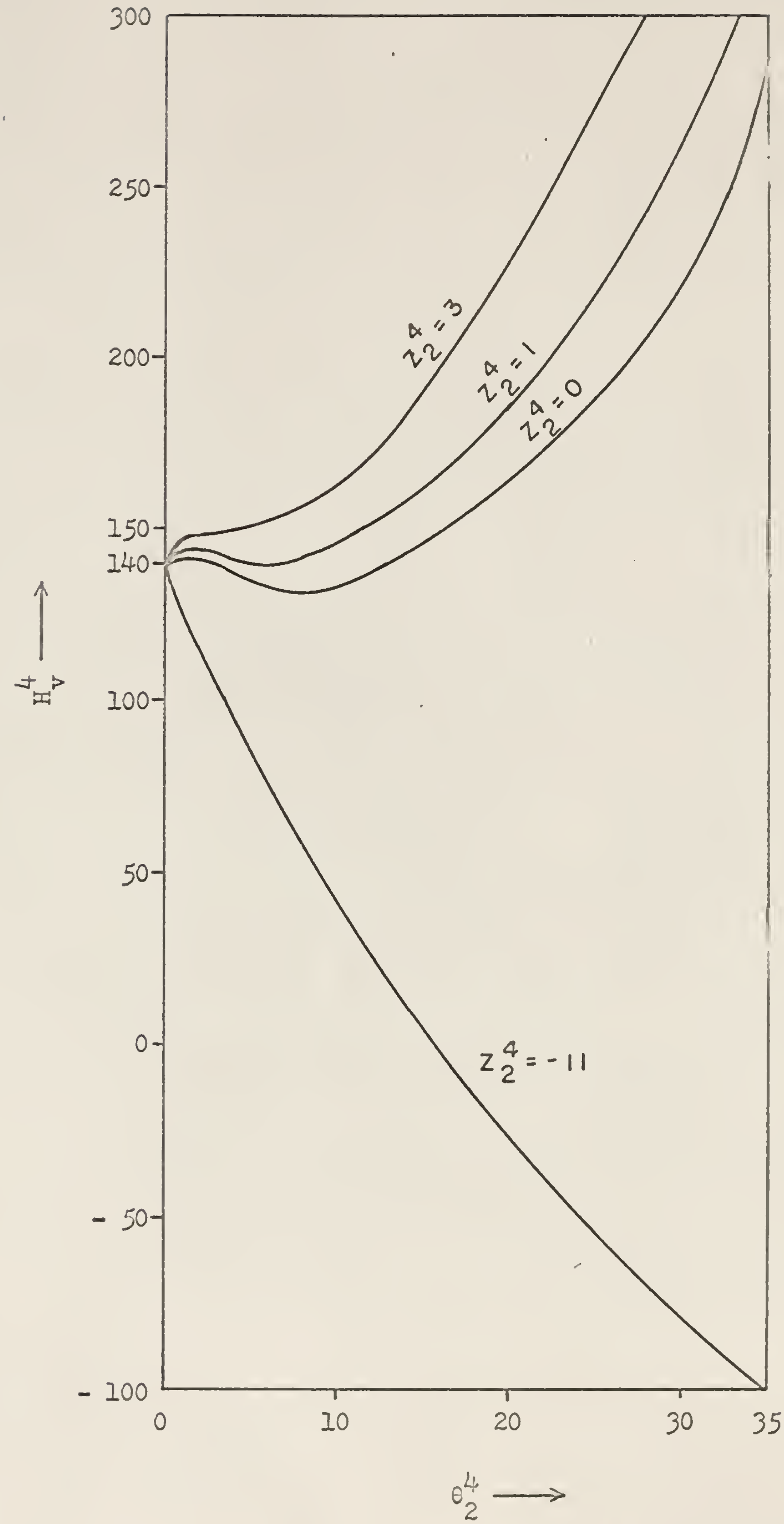

Fig. 7. Relation between H_v^4 and z_2^4 when $\theta_1^4 = 0$ and θ_2^4 varies from 0 to 35.

Following are the twelve conditions at which H_v^4 may be minimum.

(a) $H_v^4 = \text{min. at } \theta_1^4 = 0 \;\&\; \theta_2^4 = 0$ when $z_1^4 > 1 \;\&\; z_2^4 > 1$

(b) $H_v^4 = \text{min. at } \theta_1^4 = 0 \;\&\; \theta_2^4 = 0 \text{ or } 5$ when $z_1^4 > 1 \;\&\; z_2^4 = 1$

(c) $H_v^4 = \text{min. at } \theta_1^4 = 0 \;\&\; \theta_2^4 = 7.5 - 2.5\, z_2^4$ when $z_1^4 > 1 \;\&\; -11 \leq z_2^4 < 1$

(d) $H_v^4 = \text{min. at } \theta_1^4 = 0 \;\&\; \theta_2^4 = 35$ when $z_1^4 > 1 \;\&\; z_2^4 < -11$

(e) $H_v^4 = \text{min. at } 0 \leq \theta_1^4 \leq 35 \;\&\; \theta_2^4 = 0$ when $z_1^4 = 1 \;\&\; z_2^4 > 1$

(f) $H_v^4 = \text{min. at } 0 \leq \theta_1^4 \leq 35 \;\&\; \theta_2^4 = 0 \text{ or } 5$ when $z_1^4 = 1 \;\&\; z_2^4 = 1$

(g) $H_v^4 = \text{min. at } 0 \leq \theta_1^4 \leq 35 \;\&\; \theta_2^4 = -7.5 - 2.5\, z_2^4$ when $z_1^4 = 1 \;\&\; -11 \leq z_2^4 < 1$

(h) $H_v^4 = \text{min. at } 0 \leq \theta_1^4 \leq 35 \;\&\; 0 \leq \theta_2^4 \leq 35$ when $z_1^4 = 1 \;\&\; z_2^4 < -11$

(i) $H_v^4 = \text{min. at } \theta_1^4 = 35 \;\&\; \theta_2^4 = 0$ when $z_1^4 < 1 \;\&\; z_2^4 > 1$

(j) $H_v^4 = \text{min. at } 0 \leq \theta_1^4 \leq 35 \;\&\; \theta_2^4 = 0 \text{ or } 5$ when $z_1^4 < 1 \;\&\; z_2^4 = 1$

(k) $H_v^4 = \text{min. at } 0 \leq \theta_1^4 \leq 35 \;\&\; \theta_2^4 = 7.5 - 2.5\, z_2^4$ when $z_1^4 < 1 \;\&\; -11 \leq z_2^4 < 1$

(l) $H_v^4 = \text{min. at } 0 \leq \theta_1^4 \leq 35 \;\&\; 0 \leq \theta_2^4 \leq 35$ when $z_1^4 < 1 \;\&\; z_2^4 < -11$

The conditions for all H_v^n to be minimum are summarized in Table 32.

The values of z_1^n and z_2^n in Table 32 may be defined as boundary values. By systematic search of each combination of the interior and/or the boundary values of z_1^n and z_2^n for the feasible solutions, cases which do not satisfy the constraints are eliminated. The procedure is as follows:

Consider first the value of $z_1^n = 6$ and the combination of all the values of z_2^n in steps to find the feasible solutions.

Table 32. Conditions necessary for H_v^4 to be minimum for Example (7)

n	minima occur at			
	θ_1^n	θ_2^n	z_1^n	z_2^n
1	0	0	> 6	> 3.82
	0	25	> 6	= 3.82
	0	25	> 6	< 3.82
	$0 \leq \theta_1^1 \leq 25$	0	= 6	> 3.82
	$0 \leq \theta_1^1 \leq 25$	0 or 25	= 6	= 3.82
	$0 \leq \theta_1^1 \leq 25$	$0 \leq \theta_2^1 \leq 25$	= 6	< 3.82
	25	0	< 6	> 3.82
	25	0	< 6	= 3.82
	$0 \leq \theta_1^1 \leq 25$	$0 \leq \theta_2^1 \leq 25$	< 6	< 3.82
2	0	0	> .975	> - 1.1
	0	$0 \leq \theta_2^2 \leq 40$	> .975	= - 1.1
	0	40	> .975	< - 1.1
	0 or 40	0	= .975	> - 1.1
	0 or 40	$0 \leq \theta_2^2 \leq 40$	= .975	= - 1.1
	0	40	= .975	< - 1.1
	40	0	< .975	> - 1.1
	$0 \leq \theta_1^2 \leq 40$	$0 \leq \theta_2^2 \leq 40$	< .975	= - 1.1
	$0 \leq \theta_1^2 \leq 40$	$0 \leq \theta_2^2 \leq 40$	< .975	< - 1.1

Table 32. (continued)

n	minima occur at			
	θ_1^n	θ_2^n	z_1^n	z_2^n
3	0	0	$> -\frac{43}{15}$	$> -\frac{5}{6}$
	0	0 or 30	$> -\frac{43}{15}$	$= -\frac{5}{6}$
	0	30	$> -\frac{43}{15}$	$< -\frac{5}{6}$
	0 or 30	0	$= -\frac{43}{15}$	$> -\frac{5}{6}$
	0 or 30	0 or 30	$= -\frac{43}{15}$	$= -\frac{5}{6}$
	0	30	$= -\frac{43}{15}$	$< -\frac{5}{6}$
	30	0	$< -\frac{43}{15}$	$> -\frac{5}{6}$
	30	0	$< -\frac{43}{15}$	$= -\frac{5}{6}$
	$0 \leq \theta_1^3 \leq 30$	$0 \leq \theta_2^3 \leq 30$	$< -\frac{43}{15}$	$< -\frac{5}{6}$
4	0	0	> 1	> 1
	0	0 or 30	> 1	$= 1$
	0	$7.5 - 2.5\, z_2^4$	> 1	$-11 \leq z_2^4 < 1$
	0	35	> 1	< -11
	$0 \leq \theta_1^4 \leq 35$	0	$= 1$	> 1
	$0 \leq \theta_1^4 \leq 35$	0 or 5	$= 1$	$= 1$
	$0 \leq \theta_1^4 \leq 35$	$7.5 - 2.5\, z_2^4$	$= 1$	$-11 \leq z_2^4 < 1$
	$0 \leq \theta_1^4 \leq 35$	$0 \leq \theta_2^4 \leq 35$	$= 1$	< -11

Table 32. (continued)

n	minima occur at			
	θ_1^n	θ_2^n	z_1^n	z_2^n
4	35	0	< 1	> 1
	$0 \leq \theta_1^4 \leq 35$	0 or 5	< 1	$= 1$
	$0 \leq \theta_1^4 \leq 35$	$7.5 - 2.5\, z_2^4$	< 1	$- 11 \leq z_2^4 < 1$
	$0 \leq \theta_1^4 \leq 35$	$0 \leq \theta_2^4 \leq 35$	< 1	$< - 11$

Take $z_1^n = 6$ and $z_2^n = 3.82$. This does not give the feasible solution as $z_1^n = 6$ does not satisfy the constraint $\sum_{n=1}^{4} \theta_1^n = W_1$. Hence, any combination of this value of z_1^n with z_2^n will not give feasible solutions.

Next, consider the value of $z_1^n = 1$.

The value of $z_1^n = 1$ with the values of z_2^n as 3.82, 1 and $- \frac{5}{6}$ does not give feasible solutions.

The values of $z_1^n = 1$ and z_2^n in the region $- 1.1 < z_2^n < - \frac{5}{6}$ give the feasible solution shown by Table 33. This solution has four undecided control variables.

The values of $z_1^n = 1$ and $z_2^n = - 1.1$ also give a feasible solution and is shown by Table 34. This solution has five undecided control variables. No more combinations with $z_1^n = 1$ will be considered as they would give an increasing number of undecided control variables. Similarly, the values of $z_1^n = 0.975$ and $z_2^n = 3.82$ do not give a feasible solution.

The feasible solution for the combination $z_1^n = 0.975$ and $z_2^n = 1$ is presented by Table 35. This solution has five undecided control variables.

The feasible solution for the combination of $z_1^n = 0.975$ and $z_2^n = - \frac{5}{6}$

is shown by Table 36. This solution has five undecided control variables.

The other combinations of z_1^n and z_2^n will have an increasing number of undecided control variables and hence will not be considered.

Table 33. θ_i^n corresponding to the values of $z_1^n = 1$ and z_2^n in the region $-1.1 < z_2^n < -\frac{5}{6}$

n \ i	1	2	3	D^n
1	(25) $0 \le \theta_1^4 \le 25$	(0) $0 \le \theta_2^4 \le 25$	(0)	25
2	0	0	40	40
3	0	30	(0)	30
4	(15) $0 \le \theta_1^4 \le 35$	(0) $7.5 - 2.5\, z_2^4$	(20)	35
W_i	40	30	60	130

The total cost for the feasible solution of Table 33 is $360.00.

Table 34. θ_i^n corresponding to the values of $z_1^n = 1$ and $z_2^n = -1.1$

n \ i	1	2	3	D^n
1	(25) $0 \le \theta_1^1 \le 25$	(0) $0 \le \theta_2^1 \le 25$	(0)	25
2	0	(0) $0 \le \theta_2^2 \le 40$	(40)	40
3	0	30	(0)	30
4	(15) $0 \le \theta_1^4 \le 35$	(0) $7.5 - 2.5\, z_2^4$	(20)	35
W_i	40	30	60	130

The total cost for the feasible solution shown in Table 34 is \$360.00.

Table 35. θ_i^n corresponding to the values of $z_1^n = 0.975$ and $z_2^n = 1$.

n \ i	1	2	3	D^n
1	(0) $0 \le \theta_1^1 \le 25$	(25) $0 \le \theta_2^1 \le 25$	(0)	25
2	(40) 0 or 40	0	(0)	40
3	0	0	30	30
4	(0) $0 \le \theta_1^4 \le 35$	(5) 0 or 5	(30)	35
W_i	40	30	60	130

The total cost for the feasible solution shown in Table 35 is \$360.50.

Table 36. θ_i^n corresponding to the values of $z_1^n = 0.975$ and $z_2^n = -\frac{5}{6}$

n \ i	1	2	3	D^n
1	(0) $0 \le \theta_1^1 \le 25$	(0) $0 \le \theta_2^1 \le 25$	(25)	25
2	(40) 0 or 40	0	(0)	40
3	0	(30) 0 or 30	(0)	30
4	(0) 0 or 40	0	(35)	35
W_i	40	30	60	130

The total cost for the feasible solution of Table 36 is $486.00.

Comparing the total costs for the feasible solutions, the optimal solution is given by Tables 33 and 34.

Checking the condition of optimality of the solution given by Tables 33 and 34 by the perturbation method results in

n \ i	1	2	3	D^n
1	25	0	0	25
2	0	1	39	40
3	0	29	1	30
4	15	0	20	35
W_i	40	30	60	130

The total cost for the above is $365.10.

Hence, the optimal solution is given by Tables 33 and 34 and the optimal cost is $360.00.

CONCLUDING REMARKS

The discrete version of maximum principle satisfies only a necessary condition, but not the sufficient condition, so it cannot pin-point the optimal solution in most of the cases.

The systematic search method eliminates the conditions which do not give feasible solutions. A feasible solution, among the few obtained by the systematic search method, having least number of undecided control variables usually gives the optimum solution. However, this is not the case with example (3). This method is still not perfect. There should exist some better methods which may be found in the future research work.

Very recently, in a paper titled "A Note on the Discrete Maximum Principle and Distribution Problems," by Charnes and Kortanek [12] citing a simple linear cost function example, commented that the maximum principle has a cumbersome computational approach for finding an optimal solution. In their words, it runs as "the number of choices (and indeterminates) builds up at a combinatorial rate as the number of depots, which is one greater than the number of z_i's, and the number of destinations, which equal the number of Hamiltonians or sets of θ_i^n's, increases. Thus, a great deal more than direct application of the discrete maximum principle is required for effective solution. In addition to this, we may point out, using this tiny example, another serious difficulty with numerical procedures --- obtaining an optimum requires the <u>exact</u> values for the z_i, i.e., missing the correct values by however small an amount can yield the wrong values for the θ_i^n."

This is investigated using the commentator's own example (Example (2))

by the systematic search method. There is only one feasible solution to their particular example and, of course, this is the optimum one. It appears that the criticism is premature at this early stage of development of this method. Compared to established linear and non-linear programming techniques, this is in its infancy and needs future developments before it can be compared for efficiency.

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THE APPLICATION OF THE DISCRETE MAXIMUM PRINCIPLE TO TRANSPORTATION PROBLEMS WITH LINEAR AND NON-LINEAR COST FUNCTIONS

by

JAGDISHCHANDRA MAGANLAL PANCHAL

B. E. (mech.) Maharaja Sayajirao University
Baroda, India, 1959

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Optimization of transportation problems, with only one type of resource and with its total supply equal to the total demand, is studied through the application of the discrete version of maximum principle. An outline of general algorithm of the discrete version of maximum principle and formulation of the transportation problem in terms of this algorithm are given.

The purpose of this report is to present an elegant stepwise approach to solve the transportation problems by the application of the discrete version of the maximum principle. An algorithm based on this principle reduces such problems to a standard form from which a number of feasible solutions are obtainable. A systematic search method is developed to obtain feasible solutions and to find an optimum solution or solutions among the feasible solutions.

A feasible solution, among few obtained by the systematic search method, having least number of undecided control variables, usually gives the optimum one. However, this is not the case with Example (3).

Simple problems involving linear cost functions with two and three depots are systematically analysed in order to obtain generalized computational procedure for solving problems of more than three depots. A problem with four depots is solved using this procedure. Problems involving non-linear cost functions, with and without set-up costs, having two and three depots, are systematically analysed.

A very recent comment, on the application of the maximum principle to linear cost function transportation problems, that it is a cumbersome computational approach for an optimum solution is investigated by solving

the commentator's example, Example (2), by the systematic search method. It appears that the criticism is premature at this early stage of the development of this method.